



An effective version of definability in metric structures

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ABSTRACT: In this paper, a computably definable predicate in metric structures is defined and characterized. Then, it is proved that every separable infinite-dimensional Hilbert structure in an effectively presented language is computable. Moreover, every definable predicate in these structures is computable.

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1. Introduction

The usual first-order logic is not a suitable framework for mathematical structures such as Banach spaces, Banach lattices, C^* -algebras, Hilbert spaces, and etc. Logic for metric structures was first studied in the 1960s; then discontinued [4]. After that, some efforts in recent years were carried out and the following approaches appeared:

1. The logic of the positive bounded formulas with an approximate semantics [8] and [9], then
2. Compact abstract theories (CAT) [1].

These attempts culminated in a new continuous version of first-order logic which is equivalent to both of the past approaches [2]. In section 2.1, this logic is briefly introduced. In this new framework, for a metric structure \mathcal{M} and $A \subseteq M$, a definable predicate in \mathcal{M} over A is one which is approximated by a sequence of formulas in the language. Likewise, a closed set $D \subseteq M^n$ is definable in \mathcal{M} over A , if the distance predicate $d(x, D)$ is definable in \mathcal{M} over A . Note that in first-order logic, a set is definable if it is the set of all the elements defined by a first-order formula.

In section 2.2, one of the approaches of computable analysis, TTE, is explained. TTE is introduced to study the effectiveness of definability in metric structures in this paper. Computable analysis is a branch of computability theory studying the functions defined on real numbers. Type-two theory of effectivity, TTE, is based on the definitions of computable real numbers and functions by A. Turing [15], A. Grzegorzczuk [7], and D.Lacombe [11]. In this framework first, computability on finite and infinite sequences of symbols of an alphabet are defined. Then, the computability on these sequences can be transferred to other sets by using them as names [16]. This approach may be used to study computable versions of problems and theorems in analysis in a mathematical style. Also, since metric model theory is the logic of metric structures, and the relations and functions in this logic are uniformly

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continuous, TTE is a suitable approach to study effective versions of problems in metric model theory.

In section 2.3, an implementation of TTE to study the effectiveness of metric model theory is expressed. These definitions first appeared in [13]. Also, similar approach can be found in [12].

In section 3, first, a computably definable predicate is defined. Then, an effective version of a basic theorem in definability in metric structures is presented. This theorem says that a predicate P is computably definable iff there are a (δ, ρ) -computable function $u : [0, 1]^{\mathbb{N}} \rightarrow [0, 1]$ and computable L -formulas $(\psi_l(x) \mid l \in \mathbb{N})$ such that for all $a \in M^k$, $P(a) = u(\psi_l^{\mathcal{M}}(a) \mid l \in \mathbb{N})$. So, with the mathematical approach, situations in which there is an algorithm to estimate a definable predicate are characterized.

In section 4, an example is studied. Issac Goldbring [5] proved that a definable operator in a Hilbert space is of the form $\lambda I + K$, where K is a compact operator, I is the identity operator, and $\lambda \in \mathbb{R}$. In this example, first, it is proved that a separable infinite-dimensional Hilbert structure in an effectively presented language is computable. Then, every definable operator in this structure is shown to be computable.

2. Preliminaries

2.1. Metric model theory (Continuous logic)

In the following, a logic which is suitable to study metric structures is explained, [2]. Note that continuous logic is an extension of first-order logic with a discrete metric.

Assume (M, d) is a complete metric space. A metric structure \mathcal{M} based on (M, d) denoted by

$$\mathcal{M} = (M, P_i^{\mathcal{M}}, f_j^{\mathcal{M}}, c_k^{\mathcal{M}} \mid i \in I, j \in J, k \in K)$$

is defined as follows: $P^{\mathcal{M}}$, $f^{\mathcal{M}}$ and $c^{\mathcal{M}}$ are the interpretations of the predicate symbol P , the function symbol f and the constant symbol c , respectively. $P^{\mathcal{M}} : M^n \rightarrow I$ and $f^{\mathcal{M}} : M^n \rightarrow M$ are uniformly continuous, for some arity n and a bounded interval I in \mathbb{R} . Moreover, $P^{\mathcal{M}}$ and $f^{\mathcal{M}}$ are uniformly continuous with modulus Δ_P and Δ_f , respectively. Also, L consists of a real number D_L which is the diameter of (M, d) . Note that the metric d can be assumed as a binary predicate symbol and interpreted as the metric of M .

Terms are defined as in first-order logic. An atomic formula is of the form $P(t_1, \dots, t_n)$, for terms t_i and a predicate symbol P . Also, $d(t_1, t_2)$ is an atomic formula for every two terms t_1 and t_2 . Every atomic formula is a formula. Moreover, for formulas $\varphi_1, \dots, \varphi_n$ and every continuous function $u : [0, 1]^n \rightarrow [0, 1]$, $u(\varphi_1, \dots, \varphi_n)$ is a formula. And, for every formula φ and every variable x , $\sup_x \varphi$ and $\inf_x \varphi$ are formulas. Note that continuous functions u are connectives and \sup and \inf are quantifiers \forall and \exists , respectively. The interpretation of each formula without free variables, a sentence, is as usual and defined by induction. A structure \mathcal{M} is a model of a sentence φ if $\varphi^{\mathcal{M}} = 0$.

The key concept studied in this paper is definability, which is defined as follows.

Definition 2.1. Assume \mathcal{M} is a metric structure and $A \subseteq M$.

1. A predicate $P : M^n \rightarrow [0, 1]$ is *definable* in \mathcal{M} over A , if there is a sequence $(\varphi_k(x) \mid k \geq 1)$ of $L(A)$ -formulas such that

$$\forall \varepsilon > 0 \exists N \forall k \geq N \forall x \in M^n (|\varphi_k^{\mathcal{M}}(x) - P(x)| \leq \varepsilon).$$

2. A function $f : M^n \rightarrow M$ is definable in \mathcal{M} over A if and only if the function $d(f(\bar{x}), y)$ on M^{n+1} is a definable predicate in \mathcal{M} over A .
3. A set $D \subseteq M^n$ is definable in \mathcal{M} over A if the distance predicate $d(\bar{x}, D)$ is definable in \mathcal{M} over A .

The following lemmas are Theorems 2.13 and 2.15 of [10]. The Lemmes will be used in the proof of Lemma 3.6.

Lemma 2.2. Let $(a_k^n)_{k,n \in \mathbb{N}}$ be a double sequence and $\lim_{k,n \rightarrow \infty} a_k^n = a$. Then the iterated limits

$$\lim_{k \rightarrow \infty} (\lim_{n \rightarrow \infty} a_k^n), \lim_{n \rightarrow \infty} (\lim_{k \rightarrow \infty} a_k^n)$$

exist and both are equal to a if and only if

- $\lim_{n \rightarrow \infty} a_k^n$ exist for each $k \in \mathbb{N}$, and
- $\lim_{k \rightarrow \infty} a_k^n$ exist for each $n \in \mathbb{N}$.

Lemma 2.3. *If $(a_k^n)_{k,n \in \mathbb{N}}$ is a double sequence such that*

- *the iterated limit $\lim_{k \rightarrow \infty} (\lim_{n \rightarrow \infty} a_k^n) = a$, and*
- *$\lim_{n \rightarrow \infty} a_k^n$ exists uniformly in $k \in \mathbb{N}$,*

then the double limit $\lim_{k,n \rightarrow \infty} a_k^n$ exists and is equal to a .

2.2. Type-two theory of the effectivity (TTE)

In this section, the approach used to study effectivity is introduced briefly, [16]. The computability notions on natural numbers, $\mathbb{N} = \{0, 1, 2, \dots\}$ are as usual. For a fixed finite set of alphabet Σ including 0, 1, assume Σ^* is the set of words (finite sequences on Σ) and Σ^ω is the set of strings (infinite sequences on Σ). It is emphasized that this is a mathematical approach to study the computability of problems in the mathematical analysis.

Definition 2.4. A naming system on a set M is a surjective function $\nu : \subseteq X \rightarrow M$ where $X \in \{\Sigma^*, \Sigma^\omega\}$. If $X = \Sigma^*$, ν is called a notation and if $X = \Sigma^\omega$, ν is called a representation.

The following presents some examples of naming systems.

- Example 2.1.**
1. The binary notation $\nu_{\mathbb{N}} : \subseteq \Sigma^* \rightarrow \mathbb{N}$ of natural numbers is defined by $\nu_{\mathbb{N}}(a_k \dots a_0) = \sum_{i=0}^k a_i \cdot 2^i$ where $a_0, \dots, a_k \in \{0, 1\}$.
 2. A notation of integers, $\nu_{\mathbb{Z}} : \subseteq \Sigma^* \rightarrow \mathbb{Z}$ is $\nu_{\mathbb{Z}}(1w) := \nu_{\mathbb{N}}(w)$ and $\nu_{\mathbb{Z}}(0w) := -\nu_{\mathbb{N}}(w)$ for $w \in \text{dom}(\nu_{\mathbb{N}}) \setminus \{0\}$.
 3. A notation of rational numbers, $\nu_{\mathbb{Q}} : \subseteq \Sigma^* \rightarrow \mathbb{Q}$ is $\nu_{\mathbb{Q}}(\iota(u)0\iota(v)) := \frac{\nu_{\mathbb{Z}}(u)}{\nu_{\mathbb{N}}(v)}$ where $u \in \text{dom}(\nu_{\mathbb{Z}})$, $v \in \text{dom}(\nu_{\mathbb{N}})$ and $\nu_{\mathbb{N}}(v) \neq 0$.
 4. The *Cauchy representation* $\rho_C : \subseteq \Sigma^\omega \rightarrow \mathbb{R}$ is defined as follows: $\rho_C(p) = x$ if and only if there are words $w_0, w_1, \dots \in \text{dom}(\nu_{\mathbb{Q}})$ such that $p = \iota(w_0)\iota(w_1)\dots$, $|\nu_{\mathbb{Q}}(w_i) - \nu_{\mathbb{Q}}(w_k)| \leq 2^{-i}$ for $i < k$ and $x = \lim_{i \rightarrow \infty} \nu_{\mathbb{Q}}(w_i)$, which is called rapidly converges.

By the wrapping function ι and the pairing function $\langle \cdot, \cdot \rangle$, a new name can be obtained by the former ones, see Definition 2.1.7 in [16].

If there exists a naming system for a set M , a new one can be obtained for M^ω and M^k , for every $k \geq 1$.

Definition 2.5. Let $\delta : \subseteq X \rightarrow M$ be a naming system for a set M where $X \in \{\Sigma^*, \Sigma^\omega\}$. Then, so are $[\delta]^\omega$ and $[\delta]^k$ of M^ω and M^k , respectively, which are defined by

$$[\delta]^\omega(\langle p_1, p_2, \dots \rangle) := (\delta(p_1), \delta(p_2), \dots),$$

and

$$[\delta]^k(\langle p_1, p_2, \dots, p_k \rangle) := (\delta(p_1), \delta(p_2), \dots, \delta(p_k)).$$

A prefix of $p \in \Sigma^\omega$ is a finite word $w \in \Sigma^*$ such that there is a $q \in \Sigma^\omega$ with $p = wq$. Then, it is denoted by $w \sqsubseteq p$. To define a continuous and then a computable function, a topology should be set on Σ^ω which is the Cantor topology. Open sets in this topology are $w\Sigma^\omega = \{p \in \Sigma^\omega \mid w \sqsubseteq p\}$. So, the function $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ is continuous if it is continuous with respect to this topology. Also, $f : \subseteq \Sigma^* \rightarrow \Sigma^*$ is continuous with respect to the discrete topology. Note that a computable function is continuous.

In the following, a computable function on Σ^* and Σ^ω is defined, ([14], Definition 5.1 and Lemma 5.2).

- Definition 2.6.**
1. A function $f : \subseteq (\Sigma^*)^k \rightarrow \Sigma^*$ is computable if $\nu_{\mathbb{N}} \circ f \circ (\nu_{\mathbb{N}}^k)^{-1}$ is a computable function from \mathbb{N}^k into \mathbb{N} in the sense of classical computability theory.
 2. A function $h : \subseteq (\Sigma^*)^k \rightarrow \Sigma^*$ is *monotone-constant* iff

$$h(y) \downarrow \text{ and } y \sqsubseteq y' \Rightarrow h(y') \downarrow \text{ and } h(y) = h(y').$$

For monotone-constant function h , define $T_*(h) : \subseteq (\Sigma^\omega)^k \rightarrow \Sigma^*$ by

$$T_*(h)(x) = w \iff (\exists y \in (\Sigma^*)^k)(y \sqsubseteq x \wedge h(y) = w).$$

A function $f : \subseteq (\Sigma^\omega)^k \rightarrow \Sigma^*$ is Turing computable iff $f = T_*(h)$ for some Turing computable monotone-constant function $h : \subseteq (\Sigma^*)^k \rightarrow \Sigma^*$.

3. A function $h : \subseteq (\Sigma^*)^k \rightarrow \Sigma^*$ is monotone iff

$$h(y) \downarrow \text{ and } y \sqsubseteq y' \Rightarrow h(y') \downarrow \text{ and } h(y) \sqsubseteq h(y').$$

For a monotone function h define $T_\omega(h) : \subseteq (\Sigma^\omega)^k \rightarrow \Sigma^\omega$ by

$$T_\omega(h)(x) = q \iff q = \sup_{\sqsubseteq} \{h(y) \mid y \sqsubseteq x \text{ and } h(y) \downarrow\}.$$

A function $f : \subseteq (\Sigma^\omega)^k \rightarrow \Sigma^\omega$ is Turing computable iff $f = T_\omega(h)$ for some Turing computable monotone function $h : \subseteq (\Sigma^*)^k \rightarrow \Sigma^*$.

When the notion of a computable function on Σ^ω and Σ^* is established, a general computable function can be defined. This definition can be naturally extended for the functions on $M_1 \times M_2 \cdots \times M_k$.

Definition 2.7. 1. Let $\gamma : \subseteq X \rightarrow M$ and $\delta : \subseteq Y \rightarrow N$ be two naming systems where $X, Y \in \{\Sigma^*, \Sigma^\omega\}$. A function $g : \subseteq X \rightarrow Y$ is a (γ, δ) -realization of the function f if $f \circ \gamma(x) = \delta \circ g(x)$, for all $x \in \text{dom}(f \circ \gamma)$.
 2. The function $f : \subseteq M \rightarrow N$ is (γ, δ) -computable if it has a computable (γ, δ) -realization. (Figure 1)

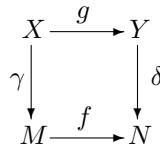


Figure 1: g is a (γ, δ) -realization of f whenever $f \circ \gamma(x) = \delta \circ g(x)$, for all $x \in X$ such that $f \circ \gamma(x)$ exists.

2.3. Effective metric model theory

In the following, the concepts of computable and decidable metric structures are explained. This approach to study the effectiveness of the metric structures was first introduced in [13].

Definition 2.8. [16]

1. An *effective metric space* is a tuple $\mathbb{M} = (M, d, A, \alpha)$ such that
 - (a) (M, d) is a separable complete metric space.
 - (b) $\alpha : \subseteq \Sigma^* \rightarrow A$ is a notation of a dense and countable subset $A \subseteq M$.
2. A *computable metric space* is an effective metric space such that
 - (a) $\text{dom}(\alpha)$ is c.e.
 - (b) $d|_{A \times A}$ is an (α, α, ρ_C) -computable function.

Similar to Example 2.1.3, a generalization of Cauchy representation can be defined for an effective metric space. This representation is defined to study the computability of functions and predicates in a metric structure.

Definition 2.9. [16] For an effective metric space $\mathbb{M} = (M, d, A, \alpha)$, the *Cauchy representation* $\delta_M : \subseteq \Sigma^\omega \rightarrow M$ is defined by $\delta_M(p) = x$, where $p = \iota(w_0)\iota(w_1)\dots$, for $w_0, w_1, \dots \in \text{dom}(\alpha)$, $d(\alpha(w_i), \alpha(w_k)) \leq 2^{-i}$ for $i < k$, and $x = \lim_{i \rightarrow \infty} \alpha(w_i)$, rapidly converges.

For instance, if we let e to be Euclidean metric over \mathbb{R} , $(\mathbb{R}, e, \mathbb{Q}, \nu_{\mathbb{Q}})$ is a computable metric space. In this case, $\delta_{\mathbb{R}}$ is exactly the Cauchy representation ρ_C in the Example 2.1.

There exists a representation η for $F^{\omega\omega}$, the set of all partial continuous functions $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ with G_δ -domain. It means $p \in \Sigma^\omega$ is a name for a continuous function $\eta_p : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ with a G_δ -domain which on input q returns the value $\eta_p(q)$. For more details of this representation, see [6] and [13].

By the above representation, a continuous function $f \in F^{\omega\omega}$ is computable if there is a computable $p \in \Sigma^\omega$ such that $f = \eta_p$.

Below, by the representation η , a new one for the set of continuous total functions $f : M_1 \rightarrow M_2$ can be obtained, for every two sets M_1 and M_2 .

Definition 2.10. [6] Let $\gamma_1 : \subseteq \Sigma^\omega \rightarrow M_1$ and $\gamma_2 : \subseteq \Sigma^\omega \rightarrow M_2$ be two representations. For the set $C(M_1, M_2)$ of continuous total functions $f : M_1 \rightarrow M_2$, define a representation $[\gamma_1 \rightarrow \gamma_2] : \subseteq \Sigma^\omega \rightarrow C(M_1, M_2)$ as follows:

$$[\gamma_1 \rightarrow \gamma_2](p) = f \iff (f \circ \gamma_1)(q) = (\gamma_2 \circ \eta_p)(q),$$

for every $q \in \Sigma^\omega$ such that $(f \circ \gamma_1)(q)$ exists.

Next, the notion of an effectively presented language L and then a computable and a decidable L -structure will be established [13].

Definition 2.11. A countable signature L is *effectively presented* if

1. The sets of variable, predicate, function and constant symbols are computable. It means if $c_V : \subseteq \Sigma^* \rightarrow Var$, $c_P : \subseteq \Sigma^* \rightarrow \mathcal{P}$, $c_F : \subseteq \Sigma^* \rightarrow \mathcal{F}$ and $c_C : \subseteq \Sigma^* \rightarrow \mathcal{C}$ are the naming systems for the sets of variables, predicate, function and constant symbols, respectively, then $dom(c_V)$, $dom(c_P)$, $dom(c_F)$ and $dom(c_C)$ are c.e subsets of Σ^* .
2. Moduli of uniform continuity of predicate and function symbols are (ρ_C, ρ_C) -computable functions.

Similar to computability theory, a notation c for $Form$, the set of L -formulas exists such that $dom(c)$ is a c.e. set. So, let $\{\varphi_n \mid n \in \mathbb{N}\}$ be an effective list of the set of all L -formulas.

Now, let (M, d, A, α) be an effective metric space. Put the Cauchy representations δ_M on M and ρ_C on $[0, 1]$. Let \mathcal{M} be a metric L -structure based on (M, d, A, α) . Assume

$$Form(\mathcal{M}, L) = \{\varphi^{\mathcal{M}} : M^{n_\varphi} \rightarrow [0, 1] \mid \varphi \text{ is an } L\text{-formula with } n_\varphi \text{ free variables}\}.$$

To define a representation on $Form(\mathcal{M}, L)$, take the representation $\beta_n = [[\delta_M]^n \rightarrow \rho_C] : \subseteq \Sigma^\omega \rightarrow Form(\mathcal{M}, L)_n$, where

$$Form(\mathcal{M}, L)_n = \{\varphi^{\mathcal{M}} : M^{n_\varphi} \rightarrow [0, 1] \mid \varphi \text{ is an } L\text{-formula with } n \text{ free variables}\},$$

for any $n \in \mathbb{N}$. Since $Form(\mathcal{M}, L) = \bigcup_{n \in \mathbb{N}} Form(\mathcal{M}, L)_n$ it follows that the function $\beta : \subseteq \Sigma^\omega \rightarrow Form(\mathcal{M}, L)$ defined by $\beta(0^n 1 p) = \beta_n(p)$ for each $p \in dom(\beta)$, is a representation for $Form(\mathcal{M}, L)$. A similar representation β_{at} can be defined for the set of all interpretations of atomic L -formulas in \mathcal{M} , $Form_{at}(\mathcal{M}, L)$, instead of the set $Form(\mathcal{M}, L)$.

Therefore, a computable and a decidable metric structure can be defined.

Definition 2.12. 1. With the preceding assumption, a metric structure \mathcal{M} is computable iff the sequence

$$(\varphi_n^{\mathcal{M}} : M^{n_\varphi} \rightarrow [0, 1] \mid \varphi \text{ is an atomic } L\text{-formula with } n_\varphi \text{ free variables})_{n \in \mathbb{N}}$$

has a computable $[\beta_{at}]^\omega$ -name.

2. Respectively, a metric structure \mathcal{M} is decidable iff the sequence

$$(\varphi_n^{\mathcal{M}} : M^{n_\varphi} \rightarrow [0, 1] \mid \varphi \text{ is an } L\text{-formula with } n_\varphi \text{ free variables})_{n \in \mathbb{N}}$$

has a computable $[\beta]^\omega$ -name.

Actually, $[\beta]^\omega$ is a naming system for $Form(\mathcal{M}, L)^\omega$ which is the set of all sequences on $Form(\mathcal{M}, L)$. Hence, for a decidable metric structure \mathcal{M} , there is an algorithm such that for a given L -formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in M$, it returns a good approximation of $\varphi^{\mathcal{M}}(a_1, \dots, a_n)$ in rational numbers. This means that, for each $\varepsilon > 0$, $r, s \in \mathbb{Q}$ is computably found such that $r < \varphi^{\mathcal{M}}(a_1, \dots, a_n) < s$ and $s - r < \varepsilon$.

3. Computably definable predicates

In this section, a computably definable predicate is defined and characterized. Let \mathcal{M} be a metric structure based on an effective metric space $\mathbb{M} = (M, d, A, \alpha)$ and assume $\rho_C = \rho$.

Definition 3.1. (Modulus of convergence) A function $e : \mathbb{N} \rightarrow \mathbb{N}$ is called a modulus of convergence of a sequence $(x_i)_{i \in \mathbb{N}}$ if for $i, k \geq e(n)$

$$|x_i - x_k| \leq 2^{-n}.$$

The following proposition is Theorem 4.2.3 of [16]. It provides a sufficient condition for the limit of a sequence to be computable.

Proposition 3.2. Let $(x_i)_{i \in \mathbb{N}}$ be a $(\nu_{\mathbb{N}}, \rho)$ -computable sequence of real numbers with computable modulus of convergence $e : \mathbb{N} \rightarrow \mathbb{N}$. Then, its limit $x = \lim_{i \rightarrow \infty} x_i$ is computable.

In the following, a computable formula is defined.

Definition 3.3. An L -formula φ with n free variables is computable in \mathcal{M} when $\varphi^{\mathcal{M}} : M^n \rightarrow [0, 1]$ is a (δ_M, ρ) -computable function.

Now, the concept of a computably definable predicate can be introduced.

Definition 3.4. A predicate $P : M^n \rightarrow [0, 1]$ (with n -arity) is computably definable in \mathcal{M} (over \emptyset) iff there is a sequence $(\varphi_k(x) \mid k \geq 1)$ of computable L -formulas such that the sequence of predicates $(\varphi_k^{\mathcal{M}}(x) : M^n \rightarrow [0, 1] \mid k \geq 1)$ is a $(\nu_{\mathbb{N}}, \rho)$ -computable sequence with a computable modulus of convergence and $P(a) = \lim_{k \rightarrow \infty} \varphi_k^{\mathcal{M}}(a)$, for every $a \in M^n$.

Obviously, if an n -arity predicate P is computably definable in \mathcal{M} then by Proposition 3.2, $P(a)$ is computable for every $a \in M^n$.

Below, Corollary 20 of [17] is presented which is a computable version of Tietze Extension Theorem. It will be used in the proof of Theorem 3.7 to characterize a computably definable predicate.

Proposition 3.5. *Every (δ, ρ) -computable function $f : \subseteq M \rightarrow \mathbb{R}$ with co-r.e domain has a (δ, ρ) -computable total extension $f : M \rightarrow \mathbb{R}$ with the same sup and inf.*

Assume

$$\mathcal{C} = \{(a_k)_{k \in \mathbb{N}} \in [0, 1]^{\mathbb{N}} \mid \forall N \in \mathbb{N} \forall i, j > N \mid a_i - a_j \mid \leq 2^{-N}\}.$$

Also, let $([0, 1]^{\mathbb{N}}, d)$ be a metric space such that the metric d is defined by

$$d((a_k), (b_k)) = \sum_{k=0}^{\infty} 2^{-k} \mid a_k - b_k \mid,$$

for every $(a_k), (b_k) \in [0, 1]^{\mathbb{N}}$. Since $([0, 1]^{\mathbb{N}}, d)$ is compact, it is separable. Therefore, let A be a countable and dense subset of $[0, 1]^{\mathbb{N}}$ and α be a notation for A . So, $\mathcal{N} = ([0, 1]^{\mathbb{N}}, d, A, \alpha)$ is an effective metric space.

Thus, the Cauchy representation δ can be defined for $[0, 1]^{\mathbb{N}}$ as follows

$$\begin{aligned} \delta(p) = (a_k)_{k \in \mathbb{N}} &: \longleftrightarrow \exists p_0, p_1, \dots \in \text{dom}(\alpha), \\ & p := \iota(p_0)\iota(p_1) \dots, \\ & d(\alpha(p_i), \alpha(p_j)) \leq 2^{-j} (i < j), \\ & (a_k)_{k \in \mathbb{N}} = \lim_{n \rightarrow \infty} \alpha(p_n). \end{aligned} \tag{1}$$

Every sequence in \mathcal{C} is Cauchy and so its limit exists in $[0, 1]$. We can define a function $f : [0, 1]^{\mathbb{N}} \rightarrow [0, 1]$ by $f((a_k)_{k \in \mathbb{N}}) = \lim_{k \rightarrow \infty} a_k$ and $\text{dom}(f) = \mathcal{C}$.

Lemma 3.6. *The above function has a closed and co-c.e domain and is (δ, ρ) -computable.*

Proof. It is obvious that \mathcal{C} is a closed and co-c.e subset of $[0, 1]^{\mathbb{N}}$. Now, let p be a δ -name of $(a_k)_{k \in \mathbb{N}}$. So, p is of the form $\iota(p_0)\iota(p_1) \dots$ such that $p_n \in \text{dom}(\alpha)$, $n \in \mathbb{N}$ and for $i > j$,

$$d(\alpha(p_i), \alpha(p_j)) \leq 2^{-j}$$

and

$$(a_k)_{k \in \mathbb{N}} = \lim_{n \rightarrow \infty} \alpha(p_n).$$

So, if $\alpha(p_n) = (q_k^n)_{k \in \mathbb{N}}$ then for every $k \in \mathbb{N}$,

$$a_k = \lim_{n \rightarrow \infty} q_k^n.$$

Thus,

$$\begin{aligned} a &= \lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} q_k^n \\ &= \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} q_k^n. \end{aligned} \tag{2}$$

The last equality is proved by Lemmas 2.2 and 2.3. Since

1. $\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} q_k^n = a$, and

2. $\lim_{n \rightarrow \infty} q_k^n = a_k$ is uniformly in $k \in \mathbb{N}$.

by lemma 2.3, $\lim_{k, n \rightarrow \infty} q_k^n = a$. And, since

1. $\lim_{k, n \rightarrow \infty} q_k^n = a$,
2. $\lim_{n \rightarrow \infty} q_k^n = a_k$, and
3. $\lim_{k \rightarrow \infty} q_k^n$ exists,

by lemma 2.2, $\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} q_k^n = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} q_k^n = a$.

The proof of the third item is as follows:

Since $a_k = \lim_{n \rightarrow \infty} q_k^n$, there exists $N_1 \in \mathbb{N}$ such that for every $n, m \geq N_1$,

$$|q_k^n - a_k| \leq 2^{-k-2}.$$

Also, $a = \lim_{k \rightarrow \infty} a_k$, for every $k < l$ implies that

$$|a_k - a_l| \leq 2^{-k-2}.$$

And, since $a_l = \lim_{n \rightarrow \infty} q_l^n$, there exists N_2 such that for every $n \geq N_2$

$$|a_l - q_l^n| \leq 2^{-l-2}.$$

Let $k < l$ and $n \geq \max\{N_1, N_2\}$. Then,

$$\begin{aligned} |q_k^n - q_l^n| &\leq |q_k^n - a_k| + |a_k - a_l| + |a_l - q_l^n| \\ &\leq 2^{-k-2} \times 2 + 2^{-l-2} \leq 3 \times 2^{-k-2} \leq 2^{-k}. \end{aligned} \tag{3}$$

The result is that $\lim_{k \rightarrow \infty} q_k^n$ is exists and for $k \leq l$, $|q_k^n - q_l^n| \leq 2^{-k}$, for every $n \in \mathbb{N}$ except finitely many numbers.

For every $n \in \mathbb{N}$, define

$$t_n = \begin{cases} q_n^n & q_n^n \in \mathbb{Q} \\ s_n & q_n^n \notin \mathbb{Q}, |s_n - q_n^n| \leq 2^{-n}, s_n \in \mathbb{Q} \end{cases}$$

So, $a = \lim_{n \rightarrow \infty} q_n^n = \lim_{n \rightarrow \infty} t_n$. If w_n is a $\nu_{\mathbb{Q}}$ -name of t_n , for every $n \in \mathbb{N}$, then $\iota(w_1)\iota(w_2)\dots$ is a computable ρ -name for a .

□

The next theorem provides necessary and sufficient conditions for a predicate is computably definable in metric structures.

Theorem 3.7. *Let M be an effective metric space. Assume $P : M^k \rightarrow [0, 1]$ is a predicate. Then, P is computably definable iff there are a (δ, ρ) -computable function $u : [0, 1]^{\mathbb{N}} \rightarrow [0, 1]$ and computable L -formulas $(\psi_l(x) \mid l \in \mathbb{N})$ such that for all $a \in M^k$, $P(a) = u(\psi_l^{\mathcal{M}}(a) \mid l \in \mathbb{N})$.*

Proof. Let P have the specified form. Then, by Prop 9.3 of [2], P is definable. Then, for every $n \in \mathbb{N}$, there is an $m_n > n$ in \mathbb{N} such that

$$|u((a_k)_{k \in \mathbb{N}}) - u((b_k)_{k \in \mathbb{N}})| \leq 2^{-n} (*)$$

whenever $a_k = b_k$ for $k = 0, \dots, m_n$. For the simplicity, let $m = m_n$. Since u is (δ, ρ) -computable, the function $u_m : [0, 1]^{m+1} \rightarrow [0, 1]$ defined by

$$u_m(a_0, \dots, a_m) := u(a_0, \dots, a_m, 0, \dots, 0, \dots)$$

is also (δ, ρ) -computable. Since u_m just accepts finite sequences, it implies u_m is $([\rho]^m, \rho)$ -computable. So, $\varphi_n(x) := u_m(\psi_0(x), \dots, \psi_m(x))$ is a $([\delta_M]^k, \rho)$ -computable, for some $k \in \mathbb{N}$. Notice that an algorithm is presented to construct this sequence. If we define $e(n) = n$ for every $n \in \mathbb{N}$, then for $i, j \geq e(n)$,

$$\begin{aligned} |\varphi_i^{\mathcal{M}}(a) - \varphi_j^{\mathcal{M}}(a)| &= |u(\psi_0^{\mathcal{M}}(a), \dots, \psi_{m_i}^{\mathcal{M}}(a), 0, \dots, 0, \dots) \\ &\quad - u(\psi_0^{\mathcal{M}}(a), \dots, \psi_{m_j}^{\mathcal{M}}(a), 0, \dots, 0, \dots)| \\ &\leq 2^{-e(n)} = 2^{-n}, \end{aligned} \tag{4}$$

according to (*). So, the modulus of uniform convergence of the sequence is computable. By proof of Prop 9.3 of [2],

$$|P(a) - \varphi_n^M(a)| \leq 2^{-n}.$$

Therefore, $P(a) = \lim_{n \rightarrow \infty} \varphi_n^M(a)$, for every $a \in M^k$. by Proposition 3.2, $P(a)$ is computable and P is a computably definable predicate.

Now, let P be computably definable. Consider the set

$$\mathcal{C} = \{(a_k)_{k \in \mathbb{N}} \in [0, 1]^{\mathbb{N}} \mid \forall N \in \mathbb{N} \forall i, j > N \mid a_i - a_j \leq 2^{-N}\}.$$

Each sequence $(a_k)_{k \in \mathbb{N}}$ in \mathcal{C} is a Cauchy sequence in $[0, 1]^{\mathbb{N}}$. So, it converges to a limit that is denoted by $\lim(a_k)$. Moreover, \mathcal{C} is a closed and co-c.e subset of $[0, 1]^{\mathbb{N}}$ and the sequence of computable formulas, $(\varphi_l^M(x) \mid l \in \mathbb{N})$ converges to $P(x)$ is in \mathcal{C} for every $x \in M^n$. According to Lemma 3.6, the function $\lim : \mathcal{C} \rightarrow [0, 1]$ is (δ, ρ) -computable. By Proposition 3.5, there is a (δ, ρ) -computable function $u : [0, 1]^{\mathbb{N}} \rightarrow [0, 1]$ that agrees with \lim on \mathcal{C} . Therefore, for every $a \in M^k$,

$$P(a) = u(\varphi_l^M(a) \mid l \in \mathbb{N}) = \lim_{l \rightarrow \infty} \varphi_l^M(a).$$

If a is $[\delta_M]^k$ -computable then $P(a)$ is ρ -computable. □

Corollary 3.8. *An operator $T : M \rightarrow M$ on an effective metric space M is computably definable if and only if there are a (δ, ρ) -computable function $u : [0, 1]^{\mathbb{N}} \rightarrow [0, 1]$ and computable L -formulas $(\psi_k(x, y) \mid k \in \mathbb{N})$ such that for all $a, b \in M$, $d(T(a), b) = u(\psi_k^M(a, b) \mid k \in \mathbb{N})$.*

Corollary 3.9. *Let \mathcal{M} be a first-order structure and $D \subseteq M^n$ is a definable set. So, there is a first-order formula φ such that $D = \{a \in M^n \mid \mathcal{M} \models \varphi(a)\}$. The structure \mathcal{M} can be assumed to be a metric structure with discrete metric d . So, D is definable in the metric structure \mathcal{M} if there is a sequence of formula $(\varphi_k(x) \mid k \in \mathbb{N})$ such that for all $a \in M^n$*

$$|\varphi_k^M(a) - d(a, D)| < \varepsilon.$$

Since d is a discrete metric, it means that there is just one formula $\varphi(x)$ which is equivalent to $d(x, D)$. Moreover, according to Theorem 3.7, $d(x, D)$ is computably definable iff there are a sequence of computable formula $\{\psi_k(x) \mid k \in \mathbb{N}\}$ and a (δ, ρ) -computable function $u : M^n \rightarrow M$ such that $d(x, D) = u(\psi_k \mid k \in \mathbb{N})$. Thus, $d(x, D)$ is computably definable in \mathcal{M} if and only if whenever one gives a computable δ_M -name of "a" then a ρ -name of $d(a, D)$ can be computed. Thus, one can computably decide whether $a \in D$ holds. So, D is a computable set in the sense of classical computability theory.

4. An example

In the following example, assume that the language is effectively presented.

A separable infinite-dimensional Hilbert space H over \mathbb{R} is a many-sorted structure

$$\mathcal{H} = ((B_n(H))_{n \geq 1}, \{I_{mn}\}_{m < n}, \{\lambda_r\}_{r \in \mathbb{R}}, +, -, \langle \rangle, 0, \{c_n\}_{n \in \mathbb{N}})$$

where

- $B_n(H) = \{x \in H : \|x\| \leq n\}$, for $n \geq 1$ where $\|x\| = \sqrt{\langle x, x \rangle}$. These sets are called domains,
- 0 is the zero vector in $B_1(H)$,
- $I_{mn} : B_m(H) \rightarrow B_n(H)$ is the inclusion map for $m < n$,
- $\lambda_r : B_n(H) \rightarrow B_{nk}(H)$ is scalar multiplication by r , for $r \in \mathbb{R}$ and $n \geq 1$ such that there is a unique integer $k \geq 1$ with $k - 1 \leq |r| < k$,
- $+, - : B_n(H) \times B_n(H) \rightarrow B_{2n}(H)$ are vector addition and subtraction,
- $\langle \rangle : B_n(H) \times B_n(H) \rightarrow [-n^2, n^2]$ is inner product for every $n \geq 1$.

- $\{c_n\}_{n \geq 1}$ is the set of constant symbols added to the structure to show the separability.

This structure is a metric structure by the metric $d(x, y) = \|x - y\|$.

The class of separable infinite-dimensional Hilbert structure is axiomatizable by the following axioms:

1. The axioms for Hilbert space,
2. The axiom for infinite dimensionality; for every $n \geq 1$,

$$\max_{1 \leq i, j \leq n} |\langle c_i, c_j \rangle - \delta_{ij}| = 0$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

3. The axiom for basis;

$$\sup_x |x - \sum_{n \geq 1} \langle x, c_n \rangle c_n| = 0$$

The last two axioms show that the structure has an orthonormal basis and so separable. Note that in the last axiom, since just finitely many coefficients are non-zero, this axiom is actually a valid sentence. The above set of axioms is denoted by SIHS. An L -structure \mathcal{M} is a model of SIHS if and only if it is isomorphic to a separable infinite-dimensional Hilbert structure.

Now let \mathcal{H} be a model of SIHS. It is obvious that the set \mathcal{A} of finite combinations of orthonormal basis with rational coefficients are a countable dense subset of H , the universe of \mathcal{H} . Let $\{e_n : n \geq 1\}$ be an orthonormal basis of H . So, \mathcal{A} can be shown by

$$\mathcal{A} = \{\sum_{n=0}^m q_{i_n} e_{i_n} : i_n \geq 1, q_{i_n} \in \mathbb{Q}, m \geq 1\}.$$

Since the language for this structure is effectively presented, there is a computable notation for $\{c_n : n \geq 1\}$. Thus, by the interpretation $c_n^{\mathcal{M}} = e_n$, for every $n \geq 1$, the set $\{e_n : n \geq 1\}$ is a computable set. Moreover, the notation $c : \Sigma^* \rightarrow \mathcal{A}$ is computable which is defined by $c(p) = \sum_{n=0}^m q_{i_n} e_{i_n}$ if and only if $p = \iota(r_{i_0}) \dots \iota(r_{i_n}) 010 \iota(p_{i_0}) \dots \iota(p_{i_n})$ such that $\nu_{\mathbb{Q}}(r_{i_j}) = q_{i_j}$ and $c_C(p_{i_j}) = c_{i_j}$ with $c_{i_j}^{\mathcal{M}} = e_{i_j}$, for $1 \leq j \leq m$.

Therefore, the Hilbert space H can be equipped with the Cauchy representation δ_H .

Now, it is proved that $(B_n(H), d|_{B_n(H)}, \mathcal{A} \cap B_n(H), c)$ is a computable metric space, for each $n \geq 1$. Note that the range of c should be limited to $B_n(H)$.

By the definition, $dom(c)$ is c.e. Therefore, it should be shown that $d|_{\mathcal{A} \times \mathcal{A}}$ is (c, c, ρ_C) -computable. It is enough to find a computable realization $f : \subseteq \Sigma^* \times \Sigma^* \rightarrow \Sigma^\omega$ such that

$$\rho_C(f(p, q)) = d_{\mathcal{A} \times \mathcal{A}}(c(p), c(q)) = \|\sum_{m=0}^{m_p} q_{i_m} e_{i_m} - \sum_{n=0}^{m_q} q_{i_n} e_{i_n}\|$$

is computable.

First, it is proved that the inner product is a computable function. The interpretation of the inner product is the function $\langle \cdot, \cdot \rangle : H^2 \rightarrow \mathbb{R}$. This function is (δ_H^2, ρ_C) -computable iff there exists a computable function $f : \subseteq \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega$ such that $\rho_C(f(p, q)) = \langle \delta_H(p), \delta_H(q) \rangle$, for every $p, q \in dom(\langle \delta_H, \delta_H \rangle)$. Let $\delta_H(p) = \lim_{k \rightarrow \infty} c(p_k) = \lim_{k \rightarrow \infty} \sum_{n=0}^{m_{p_k}} q_{i_n} e_{i_n}$ and $\delta_H(q) = \lim_{k \rightarrow \infty} c(q_k) = \lim_{k \rightarrow \infty} \sum_{m=0}^{m_{q_k}} q'_{i_m} e_{i_m}$. Then

$$\langle \delta_H(p), \delta_H(q) \rangle = \langle \lim_{k \rightarrow \infty} \sum_{n=0}^{m_{p_k}} q_{i_n} e_{i_n}, \lim_{k \rightarrow \infty} \sum_{m=0}^{m_{q_k}} q'_{i_m} e_{i_m} \rangle = \lim_{k \rightarrow \infty} \sum_{n=0}^{m_{p_k}} q_{i_n} q'_{i_n},$$

by orthonormality of the basis. So, the inner product is a computable function, since the coefficients are the rational numbers.

Therefore, the norm $\|\cdot\|$ is computable by the definition and computability of $\sqrt{\cdot}$, $-$ and $\langle \cdot, \cdot \rangle$. Also, since each e_n is the interpretation of c_n and the set of constant symbols are computable, by presenting the language effectively, the metric on $\mathcal{A} \cap B_n(H)$ is computable. Note that the coefficients are rational numbers and so computable. Therefore, $(B_n(H), d|_{B_n(H)}, \mathcal{A} \cap B_n(H), c)$ is a computable metric space, for each $n \geq 1$.

By choosing δ_n to be the Cauchy representation for $B_n(H)$, for every $n \in \mathbb{N}$, the interpretations of $+$, $-$ and I_{mn} are $([\delta_n]^2, \delta_n)$ -, $([\delta_n]^2, \delta_n)$ - and (δ_m, δ_n) -computable, respectively.

Now, it is proved that λ_r is (δ_n, δ_{nk}) -computable such that $k - 1 \leq r < k$. Let p be a computable δ_n -name for $x \in B_n(H)$ and q be a computable ρ_C -name for r . Then, $\iota \langle q(0), p(0) \rangle \iota \langle q(1), p(1) \rangle \dots$ is a computable δ_{nk} -name for $rx \in B_{nk}$.

Corollary 4.1. *In an effectively presented language, every separable infinite-dimensional Hilbert structure is computable.*

In the rest of this example, the effectiveness of definable operators which is found in [5] is studied. Let $\mathcal{H} \models \text{SIHS}$ with universe H and $T : H \rightarrow H$ be a linear operator.

Proposition 4.2. 1. *If T is a finite-rank operator then it is computably definable.*
 2. *If T is a compact operator then it is computably definable.*

Proof.

1. Assume $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis for $T(H)$. By computable version of the theorem of Fréchet-Riesz (Theorem 4.3 [3]), there exist $[\delta_H, \rho_C]$ -computable bounded linear functionals $f_1, \dots, f_n : H \rightarrow \mathbb{R}$ such that for all $x \in H$,

$$T(x) = f_1(x)e_1 + \dots + f_n(x)e_n.$$

So, for every δ_H -computable $x \in H$ and $1 \leq i \leq n$, $f_i(x)$ is ρ_C -computable. For every $1 \leq i \leq n$, by Fréchet-Riesz representation theorem $f_i(x) = \langle x, z_i \rangle$, for some $z_i \in H$. So,

$$d(T(x), y) = \sqrt{\sum_{i=1}^n (\langle x, z_i \rangle)^2 - 2\sum_{i=1}^n (\langle x, z_i \rangle \langle e_i, y \rangle) + \|y\|^2}$$

which is a formula. Since this structure is computable, the interpretation of this formula is ρ_C -computable for every δ_H -computable $x, y \in H$. Therefore, the finite-rank operator T is computably definable.

2. Let T be a compact operator and $(T_n)_{n \in \mathbb{N}}$ be a sequence of finite-rank operators such that $\|T - T_n\| \rightarrow 0$. By the first part, for every $n \in \mathbb{N}$, T_n is computably definable and moreover, it is expressed by a computable formula. For every $N \in \mathbb{N}$, there is $K \in \mathbb{N}$ such that

$$\|T(x) - y\| - \|T_K(x) - y\| \leq \|T(x) - T_K(x)\| \leq \|T - T_K\| \leq 2^{-N},$$

for x, y of suitable domains. Since $\|T_K(x) - y\|$ is a computable formula, $\|T(x) - y\|$ is a computably definable predicate. Thus, the compact operator T is computably definable. □

The following proposition is used in the proof of Theorem 4.4.

Proposition 4.3. *If T_1, T_2 are computably definable then so are $T_1 + T_2$.*

Proof. By Lemma 2.2 of [5], $T_1 + T_2$ is definable. Let $(\varphi(x, y))_{n \in \mathbb{N}}$ be a computable sequence of formulas such that $\varphi_n(a, b) \rightarrow \|T_1(a) - b\|$ rapidly converges, for a, b of suitable domains. By substituting $b - T_2(a)$ instead of b , the result is obtained. □

Therefore, it can be proved that every definable operator on a separable infinite-dimensional Hilbert structure is computable.

Theorem 4.4. *In every model of SIHS, every definable operator is computably definable.*

Proof. By Theorem 4.1 of [5], every definable operator is of the form $T = \lambda I + K$, for some compact operator K . Since the structure is computable, the scalar multiplication is computable. So, the operator λI is computably definable. Also, by Proposition 4.2.2, K is computably definable. Thus, by Proposition 4.3, T is a computably definable operator. □

5. Conclusion

In this paper, it is shown that TTE can be a powerful method to study the effectiveness of problems in Mathematical Analysis. One can use this approach to study effectiveness in a mathematical style. So, since metric model theory is the logic of studying metric structures as Hilbert spaces, TTE is a suitable way to obtain the computable versions of problems in these spaces. Moreover, by TTE, an effective version of definability in metric model theory is investigated. Also, it is proved that in an effectively presented language, every separable infinite-dimensional Hilbert structure is computable. Consequently, every definable operator on such spaces is computable.

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