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Original Article

The warped generalized Lagrange space and its application in physics

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ABSTRACT: In this paper, we define the warped generalized Lagrangian (WGL) spaces and then examine some of their properties. In the following, we generalize the "Tavakol-van den Bergh" condition in the theory of relativity(see [5]) in this space, which is an example of the application of the warped generalized Lagrangian spaces in relativity (Theorem 4.6). We show that condition EPS in these spaces holds provided that the warped function f satisfies the condition $\left(e^{2f}\right)^{i} = 0$.

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1. Preliminaries

The notion of Lagrange spaces (or Lagrange geometry) was introduced and studied by J. Kern (see [3]). We study the geometry of Lagrange spaces as a sub-geometry of the geometry of tangent bundle (TM, π, M) of a manifold M, using the principles of Analytical Mechanics given by variational problem on the integral of action of a regular Lagrangian, the law of conservation, Nöther Theorem, etc. Note that the Euler-Lagrange equations determine a canonical semi-spray S on the manifold TM. Thus we study the geometry of the warped Lagrange space using this canonical semi-spray S. In 1987, there some books were published on the Lagrange, Hamilton and the generalized Lagrange spaces (see [7], [6] and [8]).

There is not much research on the results of the warped products of Lagrange and Finsler spaces (see [1], [11] and [4]), and there is not any research on the warped product of the generalized Lagrangian spaces in our knowledge.

The Riemannian spaces class $\{\mathcal{R}^n\}$ is a subclass of the Finsler spaces class $\{\mathcal{F}^n\}$, the class $\{\mathcal{F}^n\}$ is a subclass of Lagrangian spaces class $\{\mathcal{L}^n\}$, and this is a subclass of the generalized Lagrangian spaces $\{\mathcal{GL}^n\}$. So, we have the following sequence of inclusions:

$$\{\mathcal{R}^n\} \subset \{\mathcal{F}^n\} \subset \{\mathcal{L}^n\} \subset \{\mathcal{GL}^n\}.$$

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In this paper, we show that

$$\{\mathcal{WR}^n\} \subset \{\mathcal{WF}^n\} \subset \{\mathcal{WL}^n\} \subset \{\mathcal{WGL}^n\},$$

where the letter $\mathcal{W}$ denotes the warped product.

2. Introduction

Suppose that M is a real smooth manifold with the tangent bundle (TM, π, M) and $\overset{\circ}{T}M := TM - \{0\}$.

A differentiable *Lagrangian* is a mapping

$$L : TM \to [0, \infty)$$

of class C^∞ on $\overset{\circ}{T}M$ and continuous on the null section $0 : M \to TM$ of the projection $\pi : TM \to M$.

The *Hessian* of a differentiable Lagrangian L with respect to y^i has the elements:

$$g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} \tag{1}$$

Evidently, the set of functions $g_{ij}(x, y)$ are the components of a d-tensor field, symmetric and covariant of order 2. Also, a differentiable Lagrangian L is called *regular* if:

$$rank\Big(g_{ij}(x, y)\Big) = n \ , \quad on \ \overset{\circ}{T}M. \tag{2}$$

A *Lagrange space* is a pair $L^n = (M, L(x, y))$ formed by a smooth, real n-dimensional manifold M and a regular Lagrangian $L(x, y)$ for which the d-tensor $g_{ij}(x, y)$ has a constant signature over the manifold $\overset{\circ}{T}M$. The mapping $L(x, y)$ is called the *fundamental function*, and $g_{ij}(x, y)$ the *fundamental tensor*. For example, every Riemannian manifold $(M, g_{ij}(x))$ determines a Lagrange space $L^n = (M, L(x, y))$, where

$$L(x, y) = g_{ij}(x)y^i y^j.$$

One of the most important and practical examples is the electromagnetic space.

Example 2.1. *The following Lagrangian from electrodynamics [7]*

$$L(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + \mathcal{U}(x)$$

where $\gamma_{ij}(x)$ is a pseudo-Riemannian metric, $A_i(x)$ a co-vector field and $\mathcal{U}(x)$ a smooth function on M, m, c, e begin the known constants from Physics, determine a Lagrange space $L^n = (M, L(x, y))$. It is called the Lagrange space of electrodynamics.

A first natural generalization of the notion of Lagrange space is called a *generalized Lagrange space*. R. Miron and M. Anastasiei introduced this notion (see[7] and [6]). A *generalized Lagrange space* is a pair

$$GL^n = (M, g_{ij}(x, y)),$$

where $g_{ij}(x, y)$ is a d-tensor field on the manifold $\overset{\circ}{T}M$ of type 2, symmetric, of rank n, and has a constant signature on $\overset{\circ}{T}M$. One easily sees that any Lagrange space $L^n = (M, L(x, y))$ is a generalized Lagrange space with the fundamental tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 L(x, y)}{\partial y^i \partial y^j}. \tag{3}$$

However, each space GL^n is not a Lagrange space L^n. In fact if $g_{ij}(x, y)$ is given, it may happen that the system of PDE (3) does not admits solutions in $L(x, y)$.

Theorem 2.1. *([7])*

(1) *A necessary condition in order that the system PDE (3) to admits a solution $L(x, y)$ is the d-tensor field*

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}, \tag{4}$$

be completely symmetric.

(2) *If the condition* (1) *is verified and the* $g_{ij}(x,y)$ *are* 0*-homogeneous with respect to yi, then the following function*

$$L(x,y) = g_{ij}(x,y)y^i y^j + A_i(x)y^i + \mathcal{U}(x), \tag{5}$$

is a solution of the system of PDE (3) for any arbitrary d-covector field $A_i(x)$ *and any arbitrary function* $\mathcal{U}(x)$ *on the base manifold* M*.*

The proof of the previous statement is not complicated. When the system (3) does not admit solutions in the functions $L(x,y)$, we say that the generalized Lagrange space $GL^n = (M, g_{ij}(x,y))$ is not reducible to a Lagrange space.

Remark 2.2. *The Lagrange spaces* L^n *with the fundamental function (3) give important classes of Lagrange spaces, including the Lagrange space of electrodynamics (see Example 2.1).*

Example 2.2. *(1) The pair* $GL^n = (M, g_{ij})$ *with the fundamental tensor field*

$$g_{ij}(x,y) = e^{\sigma(x,y)}\gamma_{ij}(x) \tag{6}$$

where the function $\sigma : \overset{\circ}{T}M :\rightarrow \mathbb{R}$ *is a class* C^∞ *and* $\gamma_{ij}(x)$ *is a pseudo-Riemannian metric on the manifold* M *is a generalized Lagrange space if the d-covector field* $\frac{\partial \sigma}{\partial y^i}$ *no vanishes. It is not reducible to a Lagrange space. R. Miron and R. Tavakol [9] proved that* $GL^n = (M, g_{ij}(x,y))$ *defined by (3) satisfies the Ehlers - Pirani - Schilds' axioms (or EPS axioms) of General Relativity. Also, in [10] R. Miron, R. K. Tavakol, V. Balan, and I. Roxburgh present the* Einstein and Maxwell equations *for the generalized Lagrange space (6) and characterize the case of vanishing mixed curvature tensor field of the canonical linear d-connection.*

(2) The pair $GL^n = (M, g_{ij}(x,y))$ *with*

$$g_{ij}(x,y) = \gamma_{ij}(x) + \left(1 - \frac{1}{\mathbf{n}^2(x,y)} y_i y_j\right) \ , \ \ y_i = \gamma_{ij}(x)y^j \tag{7}$$

where $g_{ij}(x)$ *is a pseudo-Riemannian metric and* $\mathbf{n}(x,y) > 1$ *is a smooth function (*$\mathbf{n}$ *is a refractive index). It gives us a generalized Lagrange space* GL^n *which is not reducible to a Lagrange space. R. G. Beil calls it Miron's metric from Relativistic Optics (see [2]).*

3. The warped generalized Lagrange space (WGL)

Let $GL^m = (M, g_{ij}(x,y))$ and $\overline{GL}^n = (\overline{M}, \overline{g}_{\alpha,\beta}(u,v))$ be two generalized Lagrange spaces of dimension m and n, respectively, and $f \in C^\infty(M)$ be a positive function. We define the *warped generalized Lagrange space* as follows

$$WGL^{m+n} = (M \times_f \overline{M}, \widetilde{g}_{ab} := g_{ij} + f^2 \overline{g}_{\alpha\beta}). \tag{8}$$

For example let $(M, L(x,y))$ and $(\overline{M}, \overline{L}(u,v))$ be Lagrangian spaces and $\widetilde{L} : T\widetilde{M} \cong TM \oplus T\overline{M} \rightarrow \mathbb{R}$ is defined as follows

$$\widetilde{L}(x,u,y,v) := L(x,y) + f^2(x)\overline{L}(u,v),$$

then $(M \times_f \overline{M}, \overline{L}(x,u,y,v))$ is a WGL space whose fundamental tensor is

$$\begin{aligned}\widetilde{g}_{ab}(x,u,y,v) &= \frac{1}{2}\frac{\partial^2 \widetilde{L}(x,u,y,v)}{\partial y^i \partial y^j} + f^2(x)\frac{1}{2}\frac{\partial^2 \widetilde{L}(x,u,y,v)}{\partial v^\alpha \partial v^\beta} \\ &= \frac{1}{2}\frac{\partial^2 L(x,y)}{\partial y^i \partial y^j} + f^2(x)\frac{1}{2}\frac{\partial^2 \overline{L}(u,v)}{\partial v^\alpha \partial v^\beta}.\end{aligned}$$

Notation. In this paper, a local coordinates system in $\widetilde{M}$ is denoted by $\mathbf{x}^a = (x^i, u^\alpha)$, where (x^i) and (u^α) are local coordinates system in M and $\overline{M}$, respectively. Also, the indexes $\{i,j,\cdots\}$, $\{\alpha,\beta,\cdots\}$ and $\{a,b,\cdots\}$ run over the ranges $\{1,2,\cdots,m\}$, $\{1,2,\cdots,n\}$ and $\{1,2,\cdots,m,m+1,\cdots,m+n\}$, respectively.

Suppose $\widetilde{N}$ is a warped non-linear connection on $T\widetilde{M} = TM \oplus T\overline{M}$ whose local coefficients are $\widetilde{N} = (\widetilde{N}^i_j, \widetilde{N}^i_\beta, \widetilde{N}^\alpha_j, \widetilde{N}^\alpha_\beta)$ (see Section4). Next, $V(\overset{\circ}{T}\widetilde{M})$ kernel of the differential of the product projection map

$$\widetilde{\pi} = (\pi, \overline{\pi}) : \overset{\circ}{T}M \oplus \overset{\circ}{T}\overline{M} \rightarrow M \times_f \overline{M}$$

is known as *vertical bundle* on the tangent bundle $\overset{\circ}{T}\widetilde{M}$ is considered. Hence, we have

$$V(\overset{\circ}{T}\widetilde{M}) = span\Big\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\alpha}\Big\}.$$

So, using the coefficients of non-linear $\widetilde{N}$, the warped non-holonomic vector fields are defined as

$$\frac{\delta^*}{\delta x^i} := \frac{\partial}{\partial x^i} - \widetilde{N}_i^j \frac{\partial}{\partial y^j} - \widetilde{N}_i^\beta \frac{\partial}{\partial v^\beta}, \tag{9}$$

$$\frac{\delta^*}{\delta u^\alpha} := \frac{\partial}{\partial u^\alpha} - \widetilde{N}_\alpha^j \frac{\partial}{\partial y^j} - \widetilde{N}_\alpha^\beta \frac{\partial}{\partial v^\beta}, \tag{10}$$

which enable us to construct a warped complementary vector subbundle $H(\overset{\circ}{T}\widetilde{M})$ to $V(\overset{\circ}{T}\widetilde{M})$ in $T(\overset{\circ}{T}\widetilde{M})$ that is locally:

$$H(\overset{\circ}{T}\widetilde{M}) = span\Big\{\frac{\delta^*}{\delta x^i}, \frac{\delta^*}{\delta u^\alpha}\Big\}.$$

We call $H(\overset{\circ}{T}\widetilde{M})$ the *warped horizontal distribution* on $T(\overset{\circ}{T}\widetilde{M})$, and we have

$$T(\overset{\circ}{T}\widetilde{M}) = H(\overset{\circ}{T}\widetilde{M}) \oplus V(\overset{\circ}{T}\widetilde{M}).$$

Now, we define the following new operators (see [11]):

$$\frac{\partial^*}{\partial y^i} := \frac{\partial}{\partial y^i} + \widetilde{N}_i^\alpha \frac{\partial}{\partial v^\alpha}, \tag{11}$$

$$\frac{\partial^*}{\partial v^\alpha} := \frac{\partial}{\partial v^\alpha} + \widetilde{N}_\alpha^i \frac{\partial}{\partial y^i}, \tag{12}$$

and we put $V^*(\overset{\circ}{T}\widetilde{M}) := span\{\frac{\partial^*}{\partial y^i}, \frac{\partial^*}{\partial v^\alpha}\}$. It follows that $V^*(\overset{\circ}{T}\widetilde{M}) \cong V(\overset{\circ}{T}\widetilde{M})$, and so, the tangent bundle of $\overset{\circ}{T}\widetilde{M}$ admits the decomposition

$$T(\overset{\circ}{T}\widetilde{M}) = H(\overset{\circ}{T}\widetilde{M}) \oplus V^*(\overset{\circ}{T}\widetilde{M}). \tag{13}$$

Let us assume that $(TM \oplus_f T\overline{M}, \widetilde{\pi} = (\pi, \overline{\pi}), M \times_f \overline{M})$ is endowed with a non-linear connection $\widetilde{N}$. Then every vector field X on $M \times_f \overline{M}$ determine on unique vector field X^{h^*} on $TM \oplus_f T\overline{M}$ such that $d\widetilde{\pi}(X^{h^*}) = (d\pi, d\overline{\pi})(X^{h^*}) = X$. The vector field X^{h^*} is called the *warped horizontal lift* of $X \in \chi(TM \oplus_f T\overline{M})$. In order to derive a local representation for X^{h^*}, we put $\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial u^\alpha}\right)^{h^*} = \frac{\delta^*}{\delta x^i} + \frac{\delta^*}{\delta u^\alpha}$. Therefore, using (13) the non-linear connection $\widetilde{N}$ induces a decomposition of every $X \in TM \oplus_f T\overline{M}$ as a sum of its warped horizontal and vertical parts $X = h^*X + v^*X = X^{h^*} + X^{v^*}$. It follows that a linear connection $\widetilde{D}$ on $T\widetilde{M} = TM \oplus_f T\overline{M}$ is a d-connection if and only if one of the following conditions holds:

$$\begin{cases} (i)\ v^*\widetilde{D}_X h^*Y = 0, h^*\widetilde{D}_X v^*Y = 0, \\ (ii)\ \widetilde{D}_X Y = h^*\widetilde{D}_X h^*Y + v^*\widetilde{D}_X v^*Y, \\ (iii)\ \widetilde{D}_X h^* = 0,\ \widetilde{D}_X v^* = 0. \end{cases}$$

Here, $h^* := \frac{\delta^*}{\delta x^i} \otimes dx^i + \frac{\delta^*}{\delta u^\alpha} \otimes du^\alpha$ and $v^* := \frac{\partial}{\partial y^i} \otimes \delta^* y^i + \frac{\partial}{\partial v^\alpha} \otimes \delta^* v^\alpha$ are the *warped horizontal and vertical projector* associated with $\widetilde{N}$, respectively, and $X, Y \in T\widetilde{M}$ ([11]). A tensor field T of type $(p+r, q+s, p'+r', q'+s')$ on $T\widetilde{M} = TM \oplus_f T\overline{M}$ is said to be the *warped d-tensor field* or $\widetilde{M}$-tensor field of type $\begin{bmatrix} p+p' & r+r' \\ q+q' & s+s' \end{bmatrix}$ on $T\widetilde{M}$ if

$$\begin{aligned} &T\Big(\omega_1, \cdots, \omega_p, \omega'_1, \cdots, \omega'_{p'}, X_1, \cdots, X_q, X'_1, \cdots, X'_{q'}, \\ &\quad \omega_{p+1}, \cdots, \omega_{p+r}, \omega'_{p'+1}, \cdots, \omega'_{p'+r'}, X_{q+1}, \cdots, X_{q+s}, X'_{q'+1}, \cdots, X'_{q'+s'}\Big) = \\ &T\Big(h^*\omega_1, \cdots, h^*\omega_p, h^*\omega'_1, \cdots, h^*\omega'_{p'+r'} h^*X_1, \cdots, h^*X_q, h^*X'_1, \cdots, h^*X'_{q'}, \\ &\quad v^*\omega_{p+1}, \cdots, v^*\omega_{p+r}, v^*\omega'_{p'+1}, \cdots, v^*\omega'_{p'+r'}, v^*X_{q+1}, \cdots, v^*X_{q+s}, v^*X'_{q'+1}, \cdots, v^*X'_{q'+s'}\Big). \end{aligned}$$

Locally, a warped d-tensor field of type $\begin{bmatrix} p+p' & r+r' \\ q+q' & s+s' \end{bmatrix}$ may is written in the form

$$
\begin{aligned}
T = T^{i_1\cdots i_p\alpha_1\cdots\alpha_{p'}\quad j_1\cdots j_r\beta_1\cdots\beta_{r'}}_{k_1\cdots k_q\lambda_1\cdots\lambda_{q'}\quad l_1\cdots l_s\mu_1\cdots\mu_{s'}} & \frac{\delta^*}{\delta x^{i_1}} \otimes \cdots \otimes \frac{\delta^*}{\delta x^{i_p}} \otimes \frac{\delta^*}{\delta u^{\alpha_1}} \otimes \cdots \otimes \frac{\delta^*}{\delta u^{\alpha_{p'}}} \otimes \\
& \frac{\partial^*}{\partial y^{j_1}} \otimes \cdots \otimes \frac{\partial^*}{\partial y^{j_r}} \otimes \frac{\partial^*}{\partial v^{\beta_1}} \otimes \cdots \otimes \frac{\partial^*}{\partial v^{\beta_{r'}}} \otimes \\
& dx^{k_1} \otimes \cdots \otimes dx^{k_q} \otimes du^{\lambda_1} \otimes \cdots \otimes du^{\lambda_{q'}} \otimes \\
& \delta^* y^{l_1} \otimes \cdots \otimes \delta^* y^{l_s} \otimes \delta^* v^{\mu_1} \otimes \cdots \otimes \delta^* v^{\mu_{l_s}}.
\end{aligned}
$$

Corollary 3.1. *We can associate to every warped d-connection on $T\widetilde{M}$ two new operators of h^*-covariant derivation and v^*-covariant derivation in the algebra of warped d-tensor fields of type $\begin{bmatrix} p+p' & r+r' \\ q+q' & s+s' \end{bmatrix}$ on $T\widetilde{M}$. Indeed, it is clear that any warped d-connection $\widetilde{D}$ on $T\widetilde{M}$ leads to the composition*

$$
\widetilde{D}_X Y = \widetilde{D}_{h^*X} Y + \widetilde{D}_{v^*X} Y, \ \ \forall X, Y \in \chi(T\widetilde{M} = TM \oplus_f T\overline{M}). \tag{14}
$$

We put, $\widetilde{D}^{h^*}_X Y := \widetilde{D}_{h^*X} Y$ and $\widetilde{D}^{v^*}_X Y := \widetilde{D}_{v^*X} Y$. The operators $\widetilde{D}^{h^*}$ and $\widetilde{D}^{v^*}$ are called the warped h-covariant derivation *or* h^*-covariant derivation *and the* warped v-covariant derivation *or* v^*-covariant derivation*, respectively.*

It is obvious that for every $X, Y \in \chi(T\widetilde{M})$, we have

$$
\begin{aligned}
&(i)\ v^*\widetilde{D}^{h^*}_X h^*Y = 0, \ h^*\widetilde{D}^{h^*}_X v^*Y = 0, \ v^*\widetilde{D}^{v^*}_X h^*Y = 0, \ h^*\widetilde{D}^{v^*}_X v^*Y = 0, \\
&(ii)\ \widetilde{D}^{h^*}_X Y = h^*\widetilde{D}^{h^*}_X h^*Y + v^*\widetilde{D}^{h^*}_X v^*Y, \ \widetilde{D}^{h^*}_X Y = h^*\widetilde{D}^{v^*}_X h^*Y + v^*\widetilde{D}^{v^*}_X v^*Y.
\end{aligned} \tag{15}
$$

Therefore, if $X = X^i \frac{\delta^*}{\delta x^i} + \overset{.}{X}^{\alpha} \frac{\delta^*}{\delta u^\alpha}$ and $Y = Y^i \frac{\partial^*}{\partial y^i} + \overset{.}{Y}^{\alpha} \frac{\partial^*}{\partial v^\alpha}$ then we have

$$
\begin{aligned}
&(i)\ \widetilde{D}^{h^*}_X = X^i \widetilde{D}_{\frac{\delta^*}{\delta x^i}} + \overset{.}{X}^{\alpha} \widetilde{D}_{\frac{\delta^*}{\delta u^\alpha}}, \\
&(ii)\ \widetilde{D}^{v^*}_Y = Y^i \widetilde{D}_{\frac{\partial^*}{\partial y^i}} + \overset{.}{Y}^{\alpha} \widetilde{D}_{\frac{\partial^*}{\partial v^\alpha}}.
\end{aligned} \tag{16}
$$

Now $\widetilde{D}$ be a d-connection on the tangent bundle $T\widetilde{M} = TM \oplus_f T\overline{M}$, thus there exists a unique system of functions

$$
(\widetilde{L}^a_{b\ c}) = (\widetilde{L}^i_{j\ k}, \widetilde{L}^i_{\beta\ k}, \widetilde{L}^i_{j\ \lambda}, \widetilde{L}^i_{\beta\ \lambda}, \widetilde{L}^\alpha_{j\ k}, \widetilde{L}^\alpha_{\beta\ k}, \widetilde{L}^\alpha_{j\ \lambda}, \widetilde{L}^\alpha_{\beta\ \lambda})
$$

and

$$
(\widetilde{C}^a_{b\ c}) = (\widetilde{C}^i_{j\ k}, \widetilde{C}^i_{\beta\ k}, \widetilde{C}^i_{j\ \lambda}, \widetilde{C}^i_{\beta\ \lambda}, \widetilde{C}^\alpha_{j\ k}, \widetilde{C}^\alpha_{\beta\ k}, \widetilde{C}^\alpha_{j\ \lambda}, \widetilde{C}^\alpha_{\beta\ \lambda})
$$

such that

$$
\begin{cases}
(1)\ \widetilde{D}_{\frac{\delta^*}{\delta x^k}} \frac{\delta^*}{\delta x^j} = \widetilde{L}^i_{j\ k} \frac{\delta^*}{\delta x^i} + \widetilde{L}^\alpha_{j\ k} \frac{\delta^*}{\delta u^\alpha} \\
(2)\ \widetilde{D}_{\frac{\delta^*}{\delta x^k}} \frac{\delta^*}{\delta u^\beta} = \widetilde{L}^i_{\beta\ k} \frac{\delta^*}{\delta x^i} + \widetilde{L}^\alpha_{\beta\ k} \frac{\delta^*}{\delta u^\alpha}
\end{cases}
\begin{cases}
(3)\ \widetilde{D}_{\frac{\delta^*}{\delta u^\lambda}} \frac{\delta^*}{\delta x^j} = \widetilde{L}^i_{j\ \lambda} \frac{\delta^*}{\delta x^i} + \widetilde{L}^\alpha_{j\ \lambda} \frac{\delta^*}{\delta u^\alpha} \\
(4)\ \widetilde{D}_{\frac{\delta^*}{\delta u^\lambda}} \frac{\delta^*}{\delta u^\beta} = \widetilde{L}^i_{\beta\ \lambda} \frac{\delta^*}{\delta x^i} + \widetilde{L}^\alpha_{\beta\ \lambda} \frac{\delta^*}{\delta u^\alpha}
\end{cases} \tag{17}
$$

and

$$
\begin{cases}
(1)\ \widetilde{D}_{\frac{\partial^*}{\partial y^k}} \frac{\partial^*}{\partial y^j} = \widetilde{C}^i_{j\ k} \frac{\partial^*}{\partial y^i} + \widetilde{C}^\alpha_{j\ k} \frac{\partial^*}{\partial v^\alpha} \\
(2)\ \widetilde{D}_{\frac{\partial^*}{\partial y^k}} \frac{\partial^*}{\partial v^\beta} = \widetilde{C}^i_{\beta\ k} \frac{\partial^*}{\partial y^i} + \widetilde{C}^\alpha_{\beta\ k} \frac{\partial^*}{\partial v^\alpha}
\end{cases}
\begin{cases}
(3)\ \widetilde{D}_{\frac{\partial^*}{\partial v^\lambda}} \frac{\partial^*}{\partial y^j} = \widetilde{C}^i_{j\ \lambda} \frac{\partial^*}{\partial y^i} + \widetilde{C}^\alpha_{j\ \lambda} \frac{\partial^*}{\partial v^\alpha} \\
(4)\ \widetilde{D}_{\frac{\partial^*}{\partial v^\lambda}} \frac{\partial^*}{\partial v^\beta} = \widetilde{C}^i_{\beta\ \lambda} \frac{\partial^*}{\partial y^i} + \widetilde{C}^\alpha_{\beta\ \lambda} \frac{\partial^*}{\partial v^\alpha}
\end{cases} \tag{18}
$$

The real functions $\left(\widetilde{L}^c_{a\ b}(x,u,y,v)\right)$ and $\left(\widetilde{C}^c_{a\ b}(x,u,y,v)\right)$ are called the *warped horizontal* and *vertical* (respectively) local coefficients of $\widetilde{D}$ with respect to the adapted local frame $\{\frac{\delta^*}{\delta x^i}, \frac{\delta^*}{\delta u^\alpha}, \frac{\partial^*}{\partial y^i}, \frac{\partial^*}{\partial v^\alpha}\}$.

Also, the *warped torsions* of d-connection $\widetilde{D}\Gamma = (\widetilde{L}, \widetilde{C})$ are given by

$$
\widetilde{T}^c_{a\ b} := \widetilde{L}^c_{a\ b} - \widetilde{L}^c_{b\ a}\ , \ \widetilde{S}^c_{a\ b} := \widetilde{C}^c_{a\ b} - \widetilde{C}^c_{b\ a}\ , \ \widetilde{P}^c_{a\ b} := \frac{\partial^* \widetilde{N}^c_a}{\partial \mathbf{y}^b} - \widetilde{L}^c_{a\ b}, \tag{19}
$$

where $\frac{\partial^*}{\partial \mathbf{y}^b} = (\frac{\partial^*}{\partial y^i}, \frac{\partial^*}{\partial v^\alpha})$.

For example, let

$$T = T^{ij\alpha\beta}_{kl\lambda\mu} \frac{\delta^*}{\delta x^i} \otimes \cdots \otimes \delta^* y^l \otimes \frac{\delta^*}{\delta u^\alpha} \otimes \cdots \otimes \delta^* v^\mu$$

be a warped d-tensor field of type $\begin{bmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{bmatrix}$, then its h^*-covariant and v^*-covariant derivatives are respectively;

$$\begin{cases} (1)\widetilde{D}^{h^*}_{\frac{\delta^*}{\delta x^t}} T = T^{ij\alpha\beta}_{kl\lambda\mu\overset{*}{|}_t} \frac{\delta^*}{\delta x^i} \otimes \cdots \otimes \delta^* y^l \otimes \frac{\delta^*}{\delta u^\alpha} \otimes \cdots \otimes \delta^* v^\mu \\ (2)\widetilde{D}^{h^*}_{\frac{\delta^*}{\delta u^\tau}} T = T^{ij\alpha\beta}_{kl\lambda\mu\overset{*}{|}_\tau} \frac{\delta^*}{\delta x^i} \otimes \cdots \otimes \delta^* y^l \otimes \frac{\delta^*}{\delta u^\alpha} \otimes \cdots \otimes \delta^* v^\mu, \end{cases}$$

$$\begin{cases} (3)\widetilde{D}^{v^*}_{\frac{\partial^*}{\partial y^t}} T = T^{ij\alpha\beta}_{kl\lambda\mu\overset{*}{\|}_t} \frac{\delta^*}{\delta x^i} \otimes \cdots \otimes \delta^* y^l \otimes \frac{\delta^*}{\delta u^\alpha} \otimes \cdots \otimes \delta^* v^\mu \\ (4)\widetilde{D}^{v^*}_{\frac{\partial^*}{\partial v^\tau}} T = T^{ij\alpha\beta}_{kl\lambda\mu\overset{*}{\|}_\tau} \frac{\delta^*}{\delta x^i} \otimes \cdots \otimes \delta^* y^l \otimes \frac{\delta^*}{\delta u^\alpha} \otimes \cdots \otimes \delta^* v^\mu, \end{cases}$$

where

$$\begin{aligned} T^{ij\alpha\beta}_{kl\lambda\mu\overset{*}{|}_t} = \frac{\delta^* T^{ij\alpha\beta}_{kl\lambda\mu}}{\delta x^t} &+ T^{rj\alpha\beta}_{kl\lambda\mu}\widetilde{L}^i_{rt} + T^{ir\alpha\beta}_{kl\lambda\mu}\widetilde{L}^j_{tr} + T^{ij\gamma\beta}_{kl\lambda\mu}\widetilde{L}^\alpha_{\gamma t} + T^{ij\alpha\gamma}_{kl\lambda\mu}\widetilde{L}^\beta_{t\gamma} \\ &- T^{ij\alpha\beta}_{rl\lambda\mu}\widetilde{L}^r_{kt} - T^{ij\alpha\beta}_{kr\lambda\mu}\widetilde{L}^r_{tl} - T^{ij\alpha\beta}_{kl\gamma\mu}\widetilde{L}^\gamma_{\lambda t} - T^{ij\alpha\beta}_{kl\lambda\gamma}\widetilde{L}^\gamma_{t\mu}, \end{aligned} \tag{20}$$

$$\begin{aligned} T^{ij\alpha\beta}_{kl\lambda\mu\overset{*}{|}_\tau} = \frac{\delta^* T^{ij\alpha\beta}_{kl\lambda\mu}}{\delta u^\tau} &+ T^{rj\alpha\beta}_{kl\lambda\mu}\widetilde{L}^i_{r\tau} + T^{ir\alpha\beta}_{kl\lambda\mu}\widetilde{L}^j_{\tau r} + T^{ij\gamma\beta}_{kl\lambda\mu}\widetilde{L}^\alpha_{\gamma\tau} + T^{ij\alpha\gamma}_{kl\lambda\mu}\widetilde{L}^\beta_{\tau\gamma} \\ &- T^{ij\alpha\beta}_{rl\lambda\mu}\widetilde{L}^r_{k\tau} - T^{ij\alpha\beta}_{kr\lambda\mu}\widetilde{L}^r_{\tau l} - T^{ij\alpha\beta}_{kl\gamma\mu}\widetilde{L}^\gamma_{\lambda\tau} - T^{ij\alpha\beta}_{kl\lambda\gamma}\widetilde{L}^\gamma_{\tau\mu}, \end{aligned} \tag{21}$$

$$\begin{aligned} T^{ij\alpha\beta}_{kl\lambda\mu\overset{*}{\|}_t} = \frac{\partial^* T^{ij\alpha\beta}_{kl\lambda\mu}}{\partial y^t} &+ T^{rj\alpha\beta}_{kl\lambda\mu}\widetilde{C}^i_{rt} + T^{ir\alpha\beta}_{kl\lambda\mu}\widetilde{C}^j_{tr} + T^{ij\gamma\beta}_{kl\lambda\mu}\widetilde{C}^\alpha_{\gamma t} + T^{ij\alpha\gamma}_{kl\lambda\mu}\widetilde{C}^\beta_{t\gamma} \\ &- T^{ij\alpha\beta}_{rl\lambda\mu}\widetilde{C}^r_{kt} - T^{ij\alpha\beta}_{kr\lambda\mu}\widetilde{C}^r_{tl} - T^{ij\alpha\beta}_{kl\gamma\mu}\widetilde{C}^\gamma_{\lambda t} - T^{ij\alpha\beta}_{kl\lambda\gamma}\widetilde{C}^\gamma_{t\mu}, \end{aligned} \tag{22}$$

$$\begin{aligned} T^{ij\alpha\beta}_{kl\lambda\mu\overset{*}{\|}_\tau} = \frac{\partial^* T^{ij\alpha\beta}_{kl\lambda\mu}}{\partial v^\tau} &+ T^{rj\alpha\beta}_{kl\lambda\mu}\widetilde{C}^i_{r\tau} + T^{ir\alpha\beta}_{kl\lambda\mu}\widetilde{C}^j_{\tau r} + T^{ij\gamma\beta}_{kl\lambda\mu}\widetilde{C}^\alpha_{\gamma\tau} + T^{ij\alpha\gamma}_{kl\lambda\mu}\widetilde{C}^\beta_{\tau\gamma} \\ &- T^{ij\alpha\beta}_{rl\lambda\mu}\widetilde{C}^r_{k\tau} - T^{ij\alpha\beta}_{kr\lambda\mu}\widetilde{C}^r_{\tau l} - T^{ij\alpha\beta}_{kl\gamma\mu}\widetilde{C}^\gamma_{\lambda\tau} - T^{ij\alpha\beta}_{kl\lambda\gamma}\widetilde{C}^\gamma_{\tau\mu}, \end{aligned} \tag{23}$$

Definition 3.2. *By using (11) and (12), we notice that there exists on $T\widetilde{M} = TM \oplus T\overline{M}$ the warped vertical field*

$$\mathbb{L} := y^k \frac{\partial^*}{\partial y^k} + v^\gamma \frac{\partial^*}{\partial v^\gamma}, \tag{24}$$

which does not vanish on the manifold $\overset{\circ}{T}\widetilde{M}$ and is independent of any Riemannian metric on the base manifold $\widetilde{M} = M \times_f \overline{M}$. It is called the warped Liouville vector field.

Corollary 3.3. *Let $WGL^{m+n} = (M \times_f \overline{M}, g_{ij} + f^2\overline{g}_{\alpha\beta})$ be a WGL space. Then the tensors*

$$\widetilde{g}_{ab}(x,u,y,v) = g_{ij}(x,y) + f^2(x)\overline{g}_{\alpha\beta}(u,v)$$

are 0-homogeneous with respect to (y^i, v^α) if

$$\begin{cases} y^k \frac{\partial^* \widetilde{g}_{ab}}{\partial y^k} = y^k \frac{\partial g_{ij}}{\partial y^k} + f^2 y^k \widetilde{N}^\lambda_k \frac{\partial \overline{g}_{\alpha\beta}}{\partial v^\lambda} = 0 \\ v^\mu \frac{\partial^* \widetilde{g}_{ab}}{\partial v^\mu} = v^\mu \widetilde{N}^k_\mu \frac{\partial g_{ij}}{\partial y^k} + f^2 v^\mu \frac{\partial \overline{g}_{\alpha\beta}}{\partial v^\mu} = 0. \end{cases} \tag{25}$$

It is obvious that if the tensors $\widetilde{g}_{ab}(x,u,y,v)$ are 0-homogeneous with respect to (y^i, v^α), i.e., the system equations (25) are hold, then the tensors $g_{ij}(x,y)$ and $\overline{g}_{\alpha\beta}(u,v)$ are not necessarily 0-homogeneous with respect to y^i and v^α, respectively.

The following theorem comes immediately from Corollary 3.3.

Theorem 3.4. *If the WGL space $WGL = (M \times_f \overline{M}, g_{ij} + f^2\overline{g}_{\alpha\beta})$ is reducible to Lagrangian space, then the generalized Lagrangian spaces $GL = (M, g_{ij})$ and $\overline{GL} = (\overline{M}, \overline{g}_{\alpha\beta})$ are not necessarily reducible to a Lagrangian spaces.*

Corollary 3.5. *If $WGL = (M \times_f \overline{M}, g_{ij} + f^2\overline{g}_{\alpha\beta})$ is reducible to Lagrange space then, the function*

$$\widetilde{L}(x,u,y,v) := g_{ij}(x,y)y^iy^j + A_i(x,u)y^i \\ + f^2(x)\overline{g}_{\alpha\beta}(u,v)v^\alpha v^\beta + f^2(x)B_\alpha(x,u)v^\alpha + \Psi(x,u) \tag{26}$$

is a solution of the PDE

$$\widetilde{g}_{ab}(x,u,y,v) = \frac{1}{2}\frac{\partial^2 \widetilde{L}}{\partial y^i \partial y^j} + \frac{1}{2}\frac{\partial^2 \widetilde{L}}{\partial v^\alpha \partial v^\beta},$$

where $A_i(x,u)$ and $B_\alpha(x,u)$ are two covector fields, and $\Psi(x,u)$ is a smooth function on $M \times_f \overline{M}$.

4. Geometry of warped generalized Lagrangian space

In this section, we will consider a WGL space

$$WGL = (\widetilde{M} = M \times_f \overline{M}, \widetilde{g} = g + e^{2f}\overline{g}),$$

where $\widetilde{M}$ is a warped pseudo Riemannian manifold, $\widetilde{\gamma}_{ab}(x,u) := \gamma_{ij}(x) + e^{2f(x)}\overline{\gamma}_{\alpha\beta}(u)$ a warped metric tensor field on $\widetilde{M}$, and $f : M \to \mathbb{R}$ a function of class C^∞ on M. If we denote the Christoffel symbols of $\widetilde{M}$, M, and $\overline{M}$ by $\widetilde{\Gamma}^a_{b\ c}$, $\Gamma^i_{j\ k}$ and $\overline{\Gamma}^\alpha_{\beta\ \lambda}$, respectively. Then we have

$$\left(\widetilde{\Gamma}^a_{b\ c}\right) = \left(\widetilde{\Gamma}^i_{j\ k}, \widetilde{\Gamma}^i_{j\ \lambda}, \widetilde{\Gamma}^i_{\beta\ \lambda}, \widetilde{\Gamma}^\alpha_{j\ k}, \widetilde{\Gamma}^\alpha_{j\ \lambda}, \widetilde{\Gamma}^\alpha_{\beta\ \lambda}\right),$$

where

$$\begin{cases} \widetilde{\Gamma}^i_{j\ k}(x,u) = \Gamma^i_{j\ k}(x), \ \ \widetilde{\Gamma}^i_{j\ \lambda}(x,u) = 0, \ \ \widetilde{\Gamma}^i_{\beta\ \lambda}(x,u) = -(e^{2f(x)})^i\overline{\gamma}_{\beta\lambda}(u) \\ \widetilde{\Gamma}^\alpha_{j\ k}(x,u) = 0, \ \ \widetilde{\Gamma}^\alpha_{j\ \lambda}(x,u) = f_j(x)\delta^\alpha_\lambda, \ \ \widetilde{\Gamma}^\alpha_{\beta\ \lambda}(x,u) = \overline{\Gamma}^\alpha_{\beta\ \lambda}(u). \end{cases} \tag{27}$$

Here, $f_i := \frac{\partial f}{\partial x^i}$ and $(e^{2f})^i := \gamma^{ij}\frac{\partial e^{2f}}{\partial x^j}$.

Now, we will assume that the following axioms hold:

A1- The warped fundamental tensor field $\widetilde{g}_{ab}(x,u,y,v)$ has the form

$$\widetilde{g}_{ab}(x,u,y,v) = e^{2\sigma(x,y)}\gamma_{ij}(x) + e^{2f(x)} \cdot e^{2\rho(u,v)}\overline{\gamma}_{\alpha\beta}(u), \tag{28}$$

where $\sigma : TM \to \mathbb{R}$ and $\rho : T\overline{M} \to \mathbb{R}$ are two C^∞ functions on $\overset{\circ}{T}M$ and $\overset{\circ}{T}\overline{M}$, and continuous on the null sections of tangent bundles TM and $T\overline{M}$, respectively.

A2- The space $WGL = (M \times_f \overline{M}, \widetilde{g}_{ab}(x,u,y,v))$ is endowed with the non-linear connection $\widetilde{N}$, whose coefficients are

$$\left(\widetilde{N}^a_b\right) = \left(\widetilde{N}^i_j(x,u,y,v), \widetilde{N}^i_\beta(x,u,y,v), \widetilde{N}^\alpha_j(x,u,y,v), \widetilde{N}^\alpha_\beta(x,u,y,v)\right),$$

where

$$\begin{cases} \widetilde{N}^i_j := \widetilde{\Gamma}^i_{j\ k}y^k + \widetilde{\Gamma}^i_{j\ \lambda}v^\lambda, \ \ \widetilde{N}^i_\beta := \widetilde{\Gamma}^i_{\beta\ k}y^k + \widetilde{\Gamma}^i_{\beta\ \lambda}v^\lambda \\ \widetilde{N}^\alpha_j := \widetilde{\Gamma}^\alpha_{j\ k}y^k + \widetilde{\Gamma}^\alpha_{j\ \lambda}v^\lambda, \ \ \widetilde{N}^\alpha_\beta := \widetilde{\Gamma}^\alpha_{\beta\ k}y^k + \widetilde{\Gamma}^\alpha_{\beta\ \lambda}v^\lambda. \end{cases} \tag{29}$$

A3- The WGL space is endowed with the canonical warped metrical d-connection $WD\Gamma(\mathbf{N})$.

Corollary 4.1. *By using (27) and (29) we have*

$$\widetilde{N}^i_j = \Gamma^i_{j\ k}y^k = N^i_j(x), \ \ \widetilde{N}^i_\beta = -(e^{2f})^i\overline{\gamma}_{\beta\lambda}v^\lambda, \tag{30}$$

$$\widetilde{N}^\alpha_j = f_jv^\alpha, \ \ \widetilde{N}^\alpha_\beta = \overline{\Gamma}^\alpha_{\beta\ \lambda}v^\lambda + f_ky^k\delta^\alpha_\beta. \tag{31}$$

The important point is that axiom **A1** show that the WGL space with the warped metric tensor (28) has the same conformal structure as the warped Riemann space specified by $\gamma_{ij}(x)+e^{2(f+\rho-\sigma)}\overline{\gamma}_{\alpha\beta}(u)$. Also, axiom **A2** and (41) implies that the auto-parallel curves of the warped non-linear connection $\widetilde{N}$ with the coefficients (30) and (31) are coincident with the auto-parallel curves of the warped Riemannian space $(M\times_f\overline{M},\gamma_{ij}(x)+e^{2f}\overline{\gamma}_{\alpha\beta}(u)$. Therefore, under axioms **A1**, **A2** and **A3**, the warped generalized Lagrange space WGL represents a convenient relativistic model.

In the WGL^{m+n} space, endowed with a nonlinear connection $\widetilde{N}$ we can introduce the d-connection $\widetilde{D}\Gamma=\left(\widetilde{L}^a_{bc},\widetilde{C}^a_{bc}\right)$. The coefficients of the warped canonical metrical d-connection are given by the *warped generalized Christoffel symbols*:

$$\begin{cases}\widetilde{L}^i_{jk}:=\frac{1}{2}g^{il}\left(\frac{\delta^*g_{lj}}{\delta x^k}+\frac{\delta^*g_{il}}{\delta x^j}-\frac{\delta^*g_{jk}}{\delta x^l}\right)\\ \widetilde{L}^i_{\beta k}:=\frac{1}{2}g^{il}\left(\frac{\delta^*g_{lk}}{\delta u^\beta}\right)=:\widetilde{L}^i_{k\beta}\\ \widetilde{L}^i_{\beta\lambda}:=-\frac{1}{2}g^{il}\left(\frac{\delta^*e^{2f}\overline{g}_{\beta\lambda}}{\delta x^l}\right)\\ \widetilde{L}^\alpha_{jk}:=-\frac{1}{2}e^{-2f}\overline{g}^{\alpha\lambda}\left(\frac{\delta^*g_{jk}}{\delta u^\lambda}\right)\\ \widetilde{L}^\alpha_{\beta k}:=\frac{1}{2}e^{-2f}\overline{g}^{\alpha\lambda}\left(\frac{\delta^*e^{2f}\overline{g}_{\beta\lambda}}{\delta x^k}\right)=:\widetilde{L}^\alpha_{k\beta}\\ \widetilde{L}^\alpha_{\beta\lambda}:=\frac{1}{2}\overline{g}^{\alpha\mu}\left(\frac{\delta^*\overline{g}_{\mu\beta}}{\delta u^\lambda}+\frac{\delta^*\overline{g}_{\lambda\mu}}{\delta u^\beta}-\frac{\delta^*\overline{g}_{\beta\lambda}}{\delta u^\mu}\right)\end{cases}\tag{32}$$

and

$$\begin{cases}\widetilde{C}^i_{jk}:=\frac{1}{2}g^{il}\left(\frac{\partial^*g_{lj}}{\partial y^k}+\frac{\partial^*g_{il}}{\partial y^j}-\frac{\partial^*g_{jk}}{\partial y^l}\right)\\ \widetilde{C}^i_{\beta k}:=\frac{1}{2}g^{il}\left(\frac{\partial^*g_{lk}}{\partial v^\beta}\right)=:\widetilde{C}^i_{k\beta}\\ \widetilde{C}^i_{\beta\lambda}:=-\frac{1}{2}g^{il}\left(\frac{\partial^*e^{2f}\overline{g}_{\beta\lambda}}{\partial y^l}\right)\\ \widetilde{C}^\alpha_{jk}:=-\frac{1}{2}e^{-2f}\overline{g}^{\alpha\lambda}\left(\frac{\partial^*g_{jk}}{\partial v^\lambda}\right)\\ \widetilde{C}^\alpha_{\beta k}:=\frac{1}{2}e^{-2f}\overline{g}^{\alpha\lambda}\left(\frac{\partial^*e^{2f}\overline{g}_{\beta\lambda}}{\partial y^k}\right)=:\widetilde{C}^\alpha_{k\beta}\\ \widetilde{C}^\alpha_{\beta\lambda}:=\frac{1}{2}\overline{g}^{\alpha\mu}\left(\frac{\partial^*\overline{g}_{\mu\beta}}{\partial v^\lambda}+\frac{\partial^*\overline{g}_{\lambda\mu}}{\partial v^\beta}-\frac{\partial^*\overline{g}_{\beta\lambda}}{\partial v^\mu}\right)\end{cases}\tag{33}$$

where $g_{ij}(x,y):=e^{2\sigma(x,y)}\gamma_{ij}(x)$ and $\overline{g}_{\alpha\beta}(u,v):=e^{2\rho(u,v)}\overline{\gamma}_{\alpha\beta}(u)$.

Corollary 4.2. *By using (32), (33), and Corollary 4.1 it results that*

$$\begin{cases}\widetilde{L}^i_{jk}=\Gamma^i_{jk}+\delta^i_k\sigma_{|_j}+\delta^i_j\sigma_{|_k}-g_{kj}\dot{\sigma}^i=L^i_{jk}\\ \widetilde{L}^i_{\beta k}=\delta^i_k\left(e^{2f}\right)^l\frac{\partial\sigma}{\partial y^l}\overline{\gamma}_{\beta\mu}v^\mu\\ \widetilde{L}^i_{\beta\lambda}=-\frac{1}{2}\left(e^{2f}\right)^i e^{-2\sigma+2\rho}\overline{\gamma}_{\beta\lambda}\left(1-v^\mu\frac{\partial\rho}{\partial v^\mu}\right)\\ \widetilde{L}^\alpha_{jk}=-e^{2f}\left(e^{2f}\right)^l\frac{\partial\sigma}{\partial y^l}g_{jk}e^{-2\rho}v^\alpha\\ \widetilde{L}^\alpha_{\beta k}=\frac{\partial f}{\partial x^k}\delta^\alpha_\beta\left(1-v^\mu\frac{\partial\rho}{\partial v^\mu}\right)\\ \widetilde{L}^\alpha_{\beta\lambda}=\overline{\Gamma}^\alpha_{\beta\lambda}+\rho_{|^*_\lambda}\delta^\alpha_\beta+\rho_{|^*_\beta}\delta^\alpha_\lambda-\overline{g}_{\beta\lambda}\rho^\alpha\\ \qquad-f_jy^j\left(\frac{\partial\rho}{\partial v^\lambda}\delta^\alpha_\beta+\frac{\partial\rho}{\partial v^\beta}\delta^\alpha_\lambda-\overline{g}_{\beta\lambda}\dot{\rho}^\alpha\right)\\ \qquad=\overline{L}^\alpha_{\beta\lambda}-f_jy^j\left(\frac{\partial\rho}{\partial v^\lambda}\delta^\alpha_\beta+\frac{\partial\rho}{\partial v^\beta}\delta^\alpha_\lambda-\overline{g}_{\beta\lambda}\dot{\rho}^\alpha\right)\end{cases}$$

and

$$\begin{cases}\widetilde{C}^i_{jk}=\dot{\sigma}_k\delta^i_j+\dot{\sigma}_j\delta^i_k-\dot{\sigma}^ig_{jk}=C^i_{jk}\\ \widetilde{C}^i_{\beta k}=\widetilde{N}^j_\beta\dot{\sigma}_j\delta^i_k\\ \widetilde{C}^i_{\beta\lambda}=-e^{2f}\overline{g}_{\beta\lambda}g^{il}\widetilde{N}^\mu_l\dot{\rho}_\mu\\ \widetilde{C}^\alpha_{jk}=-e^{-2f}\overline{g}^{\alpha\mu}\widetilde{N}^l_\mu\dot{\sigma}_lg_{jk}\\ \widetilde{C}^\alpha_{\beta k}=\widetilde{N}^\lambda_k\dot{\rho}_\lambda\delta^\alpha_\beta\\ \widetilde{C}^\alpha_{\beta\lambda}=\dot{\rho}_\lambda\delta^\alpha_\beta+\dot{\rho}_\beta\delta^\alpha_\lambda-\dot{\rho}^\alpha\overline{g}_{\beta\lambda}=\overline{C}^\alpha_{\beta\lambda}\end{cases}$$

where $\sigma_{|_k} := \frac{\delta\sigma}{\delta x^i} = \frac{\partial\sigma}{\partial x^i} - N_i^j \frac{\partial\sigma}{\partial y^j}$, $\dot{\sigma}^i := g^{ik}\dot{\sigma}_k$, $\dot{\sigma}_i := \frac{\partial\sigma}{\partial y^i}$, $\dot{\rho}_\alpha := \frac{\partial\rho}{\partial v^\alpha}$, $\dot{\rho}^\alpha := \overline{g}^{\alpha\beta}\dot{\rho}_\beta$, *and* $\rho_{|^*_\lambda} := \frac{\partial\rho}{\partial u^\lambda} - \widetilde{N}^\mu_\lambda \frac{\partial\rho}{\partial v^\lambda}$.

Corollary 4.3. *Suppose the function* ρ *is* 0*-homogeneous with* v, *and* $\left(e^{2f(x)}\right)^i = 0$. *Then by using Corollaries 4.1 and 4.2, we have*

$$\widetilde{L}^i_{\beta k} = 0,\ \widetilde{L}^i_{\beta\lambda} = 0,\ \widetilde{L}^\alpha_{jk} = 0,\ \widetilde{L}^\alpha_{\beta k} = \widetilde{\Gamma}^\alpha_{\beta k},$$

and

$$\widetilde{C}^i_{\beta k} = 0,\ \widetilde{C}^i_{\beta\lambda} = 0,\ \widetilde{C}^\alpha_{jk} = 0,\ \widetilde{C}^\alpha_{\beta k} = 0.$$

In what follows, we shall deal with the warped normal d-connections $\widetilde{D}\Gamma$ which is compatible with the warped fundamental tensor of the WGL space $WGL = (\widetilde{M} = M \times_f \overline{M}, \widetilde{g} = g + e^{2f}\overline{g})$.

Proposition 4.4. *The following properties hold*

$$\begin{cases} g_{ij\overset{*}{|}_h} = \frac{\delta^* g_{ij}}{\delta x^h} - g_{ik}\widetilde{L}^k_{hj} - g_{kj}\widetilde{L}^k_{ih} = 0, \\ g_{ij\overset{*}{|}_\lambda} = \frac{\delta^* g_{ij}}{\delta u^\lambda} - g_{ik}\widetilde{L}^k_{\lambda j} - g_{kj}\widetilde{L}^k_{i\lambda} = 0, \\ \overline{g}_{\alpha\beta\overset{*}{|}_h} = \frac{\delta^* \overline{g}_{\alpha\beta}}{\delta x^h} - \overline{g}_{\alpha\mu}\widetilde{L}^\mu_{h\beta} - \overline{g}_{\mu\beta}\widetilde{L}^\mu_{\alpha h} = -2\frac{\partial f}{\partial x^h}\overline{g}_{\alpha\beta} \Rightarrow \left(e^{2f}\overline{g}_{\alpha\beta}\right)_{\overset{*}{|}_h} = 0, \\ \overline{g}_{\alpha\beta\overset{*}{|}_\lambda} = \frac{\delta^* \overline{g}_{\alpha\beta}}{\delta u^\lambda} - \overline{g}_{\alpha\mu}\widetilde{L}^\mu_{\beta\lambda} - \overline{g}_{\mu\beta}\widetilde{L}^\mu_{\alpha\mu} = 0. \end{cases} \tag{34}$$

and

$$\begin{cases} g_{ij\overset{*}{\|}_h} = \frac{\partial^* g_{ij}}{\partial y^h} - g_{ik}\widetilde{C}^k_{hj} - g_{kj}\widetilde{C}^k_{ih} = 0, \\ g_{ij\overset{*}{\|}_\lambda} = \frac{\partial^* g_{ij}}{\partial v^\lambda} - g_{ik}\widetilde{C}^k_{\lambda j} - g_{kj}\widetilde{C}^k_{i\lambda} = 0, \\ \overline{g}_{\alpha\beta\overset{*}{\|}_h} = \frac{\partial^* \overline{g}_{\alpha\beta}}{\partial y^h} - \overline{g}_{\alpha\mu}\widetilde{C}^\mu_{h\beta} - \overline{g}_{\mu\beta}\widetilde{C}^\mu_{\alpha h} = 0, \\ \overline{g}_{\alpha\beta\overset{*}{\|}_\lambda} = \frac{\partial^* \overline{g}_{\alpha\beta}}{\partial v^\lambda} - \overline{g}_{\alpha\mu}\widetilde{C}^\mu_{\beta\lambda} - \overline{g}_{\mu\beta}\widetilde{C}^\mu_{\alpha\mu} = 0. \end{cases} \tag{35}$$

Corollary 4.5. *It follows from Proposition 4.4 that the WGL space*

$$WGL = (M \times_f \overline{M}, \widetilde{g}_{ab}(x,u,y,v) = g_{ij}(x,y) + e^{2f(x)}\overline{g}_{\alpha\beta}(u,v)),$$

is the warped metrical, *i.e.,*

$$\widetilde{g}_{ab\overset{*}{|}_h} = g_{ij\overset{*}{|}_h} + \left(e^{2f}\overline{g}_{\alpha\beta}\right)_{\overset{*}{|}_h} = 0,\ \widetilde{g}_{ab\overset{*}{|}_\lambda} = g_{ij\overset{*}{|}_\lambda} + \left(e^{2f}\overline{g}_{\alpha\beta}\right)_{\overset{*}{|}_\lambda} = 0, \tag{36}$$

and

$$\widetilde{g}_{ab\overset{*}{\|}_h} = g_{ij\overset{*}{\|}_h} + \left(e^{2f}\overline{g}_{\alpha\beta}\right)_{\overset{*}{\|}_h} = 0,\ \widetilde{g}_{ab\overset{*}{\|}_\lambda} = g_{ij\overset{*}{\|}_\lambda} + \left(e^{2f}\overline{g}_{\alpha\beta}\right)_{\overset{*}{\|}_\lambda} = 0. \tag{37}$$

Theorem 4.6 (The Main Theorem). *The WGL space with the metric (28) to satisfy the EPS conditions, that is,* $\left(\widetilde{L}^a_{bc}\right) = \left(\widetilde{\Gamma}^a_{bc}\right)$ *if and only if*

1 *the function* ρ *is* 0*-homogeneous with respect to* v^μ *and* $\left(e^{2f}\right)^i = 0$ $(\Leftrightarrow \frac{\partial f}{\partial x^k}\gamma^{ik} = 0)$ *for any* $i = 1, \cdots, m$;

2 *the function* $\sigma(x,y)$ *is* h*-constant on* M, *i.e.,*

$$\sigma_{|_i} = \frac{\partial\sigma}{\partial x^i} - \Gamma(x)^k_{ij}y^j\frac{\partial\sigma}{\partial y^k} = 0; \tag{38}$$

3 *the function* $\rho(u,v)$ *is* h^**-constant on* $\widetilde{M} = M \times_f \overline{M}$, *i.e.,*

$$\rho_{|^*_\lambda} = \frac{\partial\rho}{\partial u^\lambda} - \widetilde{N}^\mu_\lambda\frac{\partial\rho}{\partial v^\lambda} = 0. \tag{39}$$

We call relationships (38) and (39) the warped Tavakol-Van den Bergh *conditions.*

Let $\mathbf{c} : t \in [0,1] \mapsto (x^i(t), u^\alpha(t), y^i(t), v^\alpha(t)) \in TM \oplus T\overline{M}$ be a curve of class C^∞ then, its tangent vector field $\frac{d\mathbf{c}}{dt}$ can be written in the following form

$$\frac{d\mathbf{c}}{dt} = \frac{dx^i}{dt}\frac{\delta^*}{\delta x^i} + \frac{du^\alpha}{dt}\frac{\delta^*}{\delta u^\alpha} + \frac{\delta^* y^i}{dt}\frac{\partial}{\partial y^i} + \frac{\delta^* v^\alpha}{dt}\frac{\partial}{\partial v^\alpha}, \tag{40}$$

where, $\delta^* y^i := dy^i + \widetilde{N}^i_j dx^j + \widetilde{N}^i_\beta du^\beta$ and $\delta^* v^\alpha := dv^\alpha + \widetilde{N}^\alpha_j dx^j + \widetilde{N}^\alpha_\beta du^\beta$ are the dual basis of the adapted basis $\left(\frac{\delta^*}{\delta x^i}, \frac{\delta^*}{\delta u^\alpha}\right)$. The curve $\mathbf{c}$ is an (warped) auto-parallel of the warped non-linear connection $\widetilde{N}$ if

$$\begin{cases} \frac{\delta^* y^i}{dt} = \frac{dy^i}{dt} + \widetilde{N}^i_j \frac{dx^j}{dt} + \widetilde{N}^i_\beta \frac{du^\beta}{dt} = 0, \quad y^i = \frac{dx^i}{dt}, \\ \frac{\delta^* v^\alpha}{dt} = \frac{dv^\alpha}{dt} + \widetilde{N}^\alpha_j \frac{dx^j}{dt} + \widetilde{N}^\alpha_\beta \frac{du^\beta}{dt} = 0, \quad v^\alpha = \frac{du^\alpha}{dt}. \end{cases} \tag{41}$$

The warped vector field $D_{d\mathbf{c}}X$, where

$$d\mathbf{c} = dx^i \frac{\delta^*}{\delta x^i} + du^\alpha \frac{\delta^*}{\delta u^\alpha} + \delta^* y^i \frac{\partial}{\partial y^i} + \delta^* v^\alpha \frac{\partial}{\partial v^\alpha}, \tag{42}$$

will be denoted by DX and will be called the *warped covariant differentiation* of $X \in \chi(T\widetilde{M})$.

Corollary 4.7. *Suppose that $\widetilde{\sigma} := \sigma \circ P_1$ and $\widetilde{\rho} := \rho \circ P_2$ are the lift of σ and ρ to $T\widetilde{M} = TM \oplus T\overline{M}$, respectively, where $P_1 : TM \oplus T\overline{M} \to TM$ and $P_2 : TM \oplus T\overline{M} \to T\overline{M}$ are projection maps. If the functions σ and ρ apply to condition the warped Tavakol-Van den Bergh, then $\widetilde{\sigma}$ and $\widetilde{\rho}$ are constant on the warped auto-parallel curves of the warped non-linear connection $\widetilde{N}$ given by (29). In particular, the warped absolute energy of the WGL space*

$$\begin{aligned} \widetilde{\varepsilon}(x,u,y,v) &:= \varepsilon(x,y) + e^{f(x)}\overline{\varepsilon}(u,v) \\ &= e^{2\sigma(x,y)}\gamma_{ij}(x)y^i y^j + e^{2f(x)}e^{2\rho(u,v)}\overline{\gamma}_{\alpha\beta}(u)v^\alpha v^\beta, \end{aligned} \tag{43}$$

is the warped h^*-constant, and consequently it is constant on the warped auto-parallel curves of the warped non-linear connection given by (29).

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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