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# A New Analytical Method to Improve Attitude Correction in Inertial Navigation Systems

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**ABSTRACT:** Inertial Navigation Systems suffer from accumulative errors due to their dead reckoning structure. Consequently, the navigation reset or realigning process of such systems is unavoidable. Realigning starts with estimating the navigation errors and correcting the navigation states. To correct the error of attitude states in the navigation reset process, different kinds of attitude correction method are used in the literature. This paper proposed an analytical attitude correction method that can calculate the error of Euler angles more precisely than the conventional method. In addition, this new approach preserves normality and orthogonality characteristics of the transformation matrix while the conventional method leads to losing both of these conditions. The proposed method expresses the error of Euler angles as functions of Euler angles and small rotation angles. The relation between the Euler angles error and the small rotation angles is nonlinear and mathematical calculation is performed to extract the explicit functions. Numerical simulations prove that the proposed method of attitude correction has the mentioned features and its performance is dominant over the conventional method.

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## 1- Introduction

Inertial Navigation Systems (INS) suffer from increasing navigation errors due to their dead-reckoning structure. Thus, in order to realign such systems, it is vital to correct the navigation states periodically [1]. Correction of navigation states is done by using external measurements including attitude, position, and velocity, called navigation aids. The above procedure is known as navigation error reset in the literature [2]. Realigning Inertial Navigation Systems usually starts with estimating navigation errors [3-5], and then the estimated errors are used to correct attitude with a conventional method. This paper introduces a new way for attitude correction with small rotation angles.

To correct the navigation errors, it is essential to focus on the source of these errors. Perturbing INS ential equations of attitude, position, and velocity provide INS navigation error equations [6]. INS error equations are usually linear because the derived perturbed equations include only first-order terms [7]. Some INS error modeling approaches for handling the significant amplitude attitude errors are reported in [8-12].

Phi angle and psi angle approaches are two different techniques for attitude error modeling. The first technique provides error equations in the true frame of navigation [13], and the latter defines the error equations in the estimated frame of navigation [14]. It is proven that both of these techniques are equivalent [13]. Navigation errors are modeled based on small rotation angles assumption. If the attitude error magnitude diverges from the small rotation angle assumption, the transformation matrix estimated by these attitude errors will drop its normality and orthogonality characteristics. Consequently, efficient error detaching would not be reachable.

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Relation of Euler angles error and small rotation angles are derived analytically in this paper. Found on this analytical relation, a new method to improve the realigning procedure in Inertial Navigation Systems is presented. Numerical simulation proved that the presented method does not affect normality and orthogonality characteristics of the matrix of transformation while the performance is dominant over the conventional algorithms.

The paper is organized in 5 sections: section 2 discusses the conventional method for correcting attitude in Inertial Navigation Systems. In section 3, a new algorithm for attitude correction in inertial navigation is introduced. Section 4 discusses the numerical simulation results, and finally, the conclusion is made in section 5.

## 2- Conventional Method for Correction of Attitude

Based on the Euler theorem, three sequential rotations result in a transformation from one coordinate system to another. Since the commutative rule is valid for small rotations, the order of rotations is not important, and it means the sequence of Euler angles can be changed. According to appendix A, a small rotation of frame $\hat{N}$ (estimated frame of navigation) with respect to frame N (true frame of navigation) is provided by rotation tensor, and it is defined in the below equation:

$$\mathbf{R}^{\hat{N}N} = \mathbf{E} + \varepsilon\mathbf{R}^{\hat{N}N} \tag{1}$$

In equation (1), identity tensor is represented by **E**, operator of perturbation is represented by $\varepsilon$, rotation tensor between N (true navigation frame) and $\hat{N}$ (estimated navigation frame) are represented by $\mathbf{R}^{\hat{N}N}$, and $\varepsilon\mathbf{R}^{\hat{N}N}$ shows the perturbed representation of $\mathbf{R}^{\hat{N}N}$. It should be noted that by N, the North-East-Down (NED) navigation frame is intended. Appendix A shows that $\varepsilon\mathbf{R}^{\hat{N}N}$ is skew-symmetric, so the vector $\varepsilon\mathbf{R}^{\hat{N}N}$ is defined as follows:

$$[\varepsilon\mathbf{r}^{\hat{N}N}]^{\hat{N}} = [\varepsilon\mathbf{r}^{\hat{N}N}]^{N} = \begin{bmatrix} \varepsilon\phi \\ \varepsilon\theta \\ \varepsilon\psi \end{bmatrix} \tag{2}$$

It should be noted that yaw, roll, and pitch angle errors differ from vector elements $\varepsilon\varphi$, $\varepsilon\theta$, and $\varepsilon\psi$, called small rotation angles. These Euler angles errors are defined as follows:

$$\begin{aligned} \Delta\psi &= \hat{\psi} - \psi \\ \Delta\theta &= \hat{\theta} - \theta \\ \Delta\phi &= \hat{\phi} - \phi \end{aligned} \tag{3}$$

where $\psi$, $\varphi$, and $\theta$ denote true Euler angles while $\hat{\psi}$, $\hat{\varphi}$, and $\hat{\theta}$ represent estimated yaw, roll, and pitch angles. According to Appendix B, the small rotation angles dynamic model is defined as follows:

$$\begin{aligned}
\varepsilon\dot{\varphi} &= \omega_d\varepsilon\lambda - \frac{v_e}{R_e^2}\varepsilon h + \frac{1}{R_e}\varepsilon v_e + (\omega_d - \frac{v_e\tan\lambda}{R_e})\varepsilon\theta + \frac{v_n}{R_e}\varepsilon\psi \\
&\quad -\cos\hat{\psi}\cos\hat{\theta}\,\delta\omega_x - (\cos\hat{\psi}\sin\hat{\theta}\sin\hat{\varphi} - \sin\hat{\psi}\cos\hat{\varphi})\delta\omega_y \\
&\quad -(\cos\hat{\psi}\sin\hat{\theta}\cos\hat{\varphi} + \sin\hat{\psi}\sin\hat{\varphi})\delta\omega_z \\
\varepsilon\dot{\theta} &= \frac{v_n}{R_e^2}\varepsilon h - \frac{1}{R_e}\varepsilon v_n + (\frac{v_e\tan\lambda}{R_e} - \omega_d)\varepsilon\varphi + (\omega_n + \frac{v_e}{R_e})\varepsilon\psi \\
&\quad -\sin\hat{\psi}\cos\hat{\theta}\,\delta\omega_x - (\sin\hat{\psi}\sin\hat{\theta}\sin\hat{\varphi} + \cos\hat{\psi}\cos\hat{\varphi})\delta\omega_y \\
&\quad -(\sin\hat{\psi}\sin\hat{\theta}\cos\hat{\varphi} - \cos\hat{\psi}\sin\hat{\varphi})\delta\omega_z \\
\varepsilon\dot{\psi} &= -(\omega_n + \frac{v_e\sec^2\lambda}{R_e})\varepsilon\lambda + \frac{v_e\tan\lambda}{R_e^2}\varepsilon h - \frac{\tan\lambda}{R_e}\varepsilon v_e - \frac{v_n}{R_e}\varepsilon\varphi - \\
&\quad (\omega_n + \frac{v_e}{R})\varepsilon\theta + \sin\hat{\theta}\,\delta\omega_x - \cos\hat{\theta}\sin\hat{\varphi}\,\delta\omega_y - \cos\hat{\theta}\cos\hat{\varphi}\,\delta\omega_z
\end{aligned} \tag{4}$$

Although equation (4) reveals the relation between estimated Euler angles and small rotation angles, expression of $\Delta\varphi$, $\Delta\theta$, $\Delta\psi$ in terms of $\varepsilon\varphi$, $\varepsilon\theta$, $\varepsilon\psi$ is not developed. This relation is the subject of section 3.

In order to align Inertial Navigation Systems, the usual procedure starts with estimating small rotation angles by a filter, which uses equation (4) as its dynamic model. Afterwards, employing equations (1-2) updates the rotation tensor $\mathbf{R}^{\hat{N}N}$, and finally attitude (Euler angles) correction will be done using the following equations [19]:

$$\begin{aligned} [T]^{\hat{N}N} &= [\bar{\mathbf{R}}^{\hat{N}N}]^{\hat{N}} \\ [T]^{BN} &= [T]^{B\hat{N}}[T]^{\hat{N}N} \end{aligned} \tag{5}$$

$$\begin{cases} \theta = \arcsin(-t_{13}) \\ \phi = \arccos(t_{33}/\cos\theta)\,\mathrm{sign}(t_{23}) \\ \psi = \arccos(t_{11}/\cos\theta)\,\mathrm{sign}(t_{12}) \end{cases} \tag{6}$$

where, $[T]^{BN} = [t_{i,j}]; (i, j = 1,2,3)$.

Resetting the transformation matrix $[T]^{BN}$ Employing equation (5) leads to numerical errors. Significantly, when the rotation angles deviate from the small rotation assumption, the transformation matrix loses its normality and orthogonality characteristics, and may eventually cause divergence of attitude propagation. A new analytical algorithm to correct the matrix of transformation without the mentioned drawbacks is introduced in the next part.

## 3- Proposed Analytical Algorithm for Correction of Attitude

This part specifies the relevance of Euler angles error and small rotation angles and a new technique for correcting attitude is obtained. This technique is advantageous to conventional algorithms. In the next section, a numerical comparison will prove this claim.

Expressing $\varepsilon\mathbf{R}^{\hat{N}N}$ in navigation frame is obtained by expanding equation (1):

$$\begin{aligned} [\varepsilon\mathbf{R}^{\hat{N}N}]^{N} &= [\mathbf{R}^{\hat{N}N}]^{N} - [\mathbf{E}]^{N} = \\ [\bar{T}]^{\hat{N}N} - [\mathbf{E}]^{N} &= [T]^{N\hat{N}} - [\mathbf{E}]^{N} \\ &= [T]^{NB}[T]^{B\hat{N}} - [\mathbf{E}]^{N} \end{aligned} \tag{7}$$

applying the transpose operator on equation (7) yields:

$$\begin{aligned} &\overline{[\varepsilon\mathbf{R}^{\hat{N}N}]^{N}} = \overline{[T]^{NB}[T]^{B\hat{N}}} - \overline{[\mathbf{E}]^{N}} \rightarrow \\ &-[\varepsilon\mathbf{R}^{\hat{N}N}]^{N} = [\bar{T}]^{B\hat{N}}[\bar{T}]^{NB} - [\mathbf{E}]^{N} \rightarrow \\ &[\varepsilon\mathbf{R}^{\hat{N}N}]^{N} = [\mathbf{E}]^{N} - [\bar{T}]^{B\hat{N}}[T]^{BN} \end{aligned} \tag{8}$$

Considering matrixes of transformation $[T]^{BN}$ and $[T]^{B\hat{N}}$ in close form, yields [16]:

$$[T]^{BN} = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix} \tag{9}$$

$$[T]^{\hat{B}\hat{N}} = \begin{bmatrix} \cos\hat{\psi}\cos\hat{\theta} & \sin\hat{\psi}\cos\hat{\theta} & -\sin\hat{\theta} \\ \cos\hat{\psi}\sin\hat{\theta}\sin\hat{\phi} - \sin\hat{\psi}\cos\hat{\phi} & \sin\hat{\psi}\sin\hat{\theta}\sin\hat{\phi} + \cos\hat{\psi}\cos\hat{\phi} & \cos\hat{\theta}\sin\hat{\phi} \\ \cos\hat{\psi}\sin\hat{\theta}\cos\hat{\phi} + \sin\hat{\psi}\sin\hat{\phi} & \sin\hat{\psi}\sin\hat{\theta}\cos\hat{\phi} - \cos\hat{\psi}\sin\hat{\phi} & \cos\hat{\theta}\cos\hat{\phi} \end{bmatrix} \quad (10)$$

Exerting equations (3), (9), and (10) into equation (8) and performing some algebraic works, outcomes are as follows:

$$[\varepsilon\mathbf{R}^{\hat{N}N}]^{N} = \begin{bmatrix} 0 & -\varepsilon\psi & \varepsilon\theta \\ \varepsilon\psi & 0 & -\varepsilon\varphi \\ -\varepsilon\theta & \varepsilon\varphi & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\Delta_3 & \Delta_2 \\ \Delta_1 & 0 & -\Delta_1 \\ -\Delta_2 & \Delta_1 & 0 \end{bmatrix}$$

$$\begin{aligned} \Delta_1 &= \Delta\theta\sin\hat{\psi} - \Delta\varphi\cos\hat{\psi}\cos\hat{\theta} - \Delta\psi\,\Delta\theta\cos\hat{\psi} - \\ &\quad \Delta\varphi\Delta\psi\cos\hat{\theta}\sin\hat{\psi} \\ \Delta_2 &= -\Delta\theta\cos\hat{\psi} - \Delta\varphi\cos\hat{\theta}\sin\hat{\psi} - \\ &\quad \Delta\varphi\Delta\theta\sin\hat{\psi}\sin\hat{\theta} \\ \Delta_3 &= \Delta\varphi\sin\hat{\theta} - \Delta\psi - \Delta\varphi\Delta\theta\cos^2\hat{\psi}\cos\hat{\theta} - \\ &\quad \Delta\varphi\Delta\psi\,\Delta\theta\cos\hat{\psi}\cos\hat{\theta}\sin\hat{\psi} \end{aligned} \quad (11)$$

Resorting Equation (11) results:

$$\begin{aligned} \varepsilon\varphi &= \Delta\theta\sin\hat{\psi} - \Delta\varphi\cos\hat{\psi}\cos\hat{\theta} - \\ &\quad \Delta\psi\,\Delta\theta\cos\hat{\psi} - \Delta\varphi\Delta\psi\cos\hat{\theta}\sin\hat{\psi} \\ \varepsilon\theta &= -\Delta\theta\cos\hat{\psi} - \Delta\varphi\cos\hat{\theta}\sin\hat{\psi} - \\ &\quad \Delta\varphi\Delta\theta\sin\hat{\psi}\sin\hat{\theta} \\ \varepsilon\psi &= \Delta\varphi\sin\hat{\theta} - \Delta\psi - \Delta\varphi\Delta\theta\cos^2\hat{\psi}\cos\hat{\theta} - \\ &\quad \Delta\varphi\Delta\psi\,\Delta\theta\cos\hat{\psi}\cos\hat{\theta}\sin\hat{\psi} \end{aligned} \quad (12)$$

Equation (12) shows that it is possible to express small rotation angles as a function of error of Euler angles ($\Delta\varphi$, $\Delta\theta$ and $\Delta\psi$) and estimated Euler angles ($\hat{\theta}$ and $\hat{\psi}$).

In integrated Inertial Navigation Systems, small rotation angles and Euler angles are estimated and calculated by filter and attitude propagation, respectively, so the error of Euler angles could be obtained by employing equation (12). Since equation (12) reveals the nonlinear relation of small rotation angles and Euler angles error, there is no way to obtain an explicit formula for calculating Euler angles error. In this situation, the authors suggest using explicit approximate equations to calculate Euler angles error, which is considered the main contribution of this study. In order to derive such explicit equations, higher-order terms ($\Delta^2$ and $\Delta^3$) are neglected, and only first-order terms are considered. The procedure to derive the explicit formula would be:

$$\begin{cases} \varepsilon\varphi = \Delta\theta\sin\hat{\psi} - \Delta\varphi\cos\hat{\psi}\cos\hat{\theta} \\ \varepsilon\theta = -\Delta\theta\cos\hat{\psi} - \Delta\varphi\cos\hat{\theta}\sin\hat{\psi} \rightarrow \\ \varepsilon\psi = \Delta\varphi\sin\hat{\theta} - \Delta\psi \end{cases}$$
$$\begin{cases} \Delta\varphi = -(\varepsilon\varphi\cos\hat{\psi} + \varepsilon\theta\sin\hat{\psi})\sec\hat{\theta} \\ \Delta\theta = \varepsilon\varphi\sin\hat{\psi} - \varepsilon\theta\cos\hat{\psi} \\ \Delta\psi = -(\varepsilon\varphi\cos\hat{\psi} + \varepsilon\theta\sin\hat{\psi})\tan\hat{\theta} \end{cases} \quad (13)$$

If only third-order terms ($\Delta^3$) are neglected, the explicit derived formula would be more accurate:

$$\begin{aligned} \varepsilon\varphi &= \Delta\theta\sin\hat{\psi} - \Delta\varphi\cos\hat{\psi}\cos\hat{\theta} - \\ &\quad \Delta\psi\,(\Delta\theta\cos\hat{\psi} + \Delta\varphi\cos\hat{\theta}\sin\hat{\psi}) \end{aligned} \quad (14)$$

$$\begin{aligned} \varepsilon\theta &= -(\Delta\theta\cos\hat{\psi} + \Delta\varphi\cos\hat{\theta}\sin\hat{\psi}) - \\ &\quad \Delta\varphi\Delta\theta\sin\hat{\psi}\sin\hat{\theta} \end{aligned} \quad (15)$$

$$\varepsilon\psi = -\Delta\psi + \Delta\phi(\sin\hat{\theta} - \Delta\theta\cos^2\hat{\psi}\cos\hat{\theta}) \quad (16)$$

The simplified form of equation (16) would be:

$$\Delta\psi = \Delta\phi(\sin\hat{\theta} - \Delta\theta\cos^2\hat{\psi}\cos\hat{\theta}) - \varepsilon\psi \quad (17)$$

Exerting equation (17) into equation (14) and eliminating the third-order ($\Delta^3$) terms yields:

$$\begin{aligned} \varepsilon\varphi &= +\Delta\theta(\sin\hat{\psi} + \cos\hat{\psi}\,\varepsilon\psi) + \\ &\quad \Delta\varphi(\cos\hat{\theta}\sin\hat{\psi}\,\varepsilon\psi - \cos\hat{\psi}\cos\hat{\theta}) \\ &\quad -\Delta\varphi^2\sin\hat{\theta}\cos\hat{\theta}\sin\hat{\psi} - \\ &\quad \Delta\varphi\Delta\theta\sin\hat{\theta}\cos\hat{\psi} \end{aligned} \quad (18)$$

Another appearance of equation (15) would be:

$$\Delta\theta = \frac{-\varepsilon\theta - \Delta\phi\cos\hat{\theta}\sin\hat{\psi}}{\cos\hat{\psi} + \Delta\phi\sin\hat{\psi}\sin\hat{\theta}} \quad (19)$$

Exerting equation (19) into equation (18), eliminating third-order ($\Delta^3$) terms, and performing some algebraic works yields:

$$a(\Delta\phi)^2 + b(\Delta\phi) + c = 0 \quad (20)$$

$$\begin{aligned} a &= \sin\hat{\theta}\cos\hat{\theta}\sin\hat{\psi}(\sin\hat{\psi}\,\varepsilon\psi - \cos\hat{\psi}) \\ b &= -\cos\hat{\theta} + \sin\hat{\theta}(\cos\hat{\psi}\,\varepsilon\theta - \sin\hat{\psi}\,\varepsilon\phi) \\ c &= -\sin\hat{\psi}\,\varepsilon\theta - \cos\hat{\psi}\,(\varepsilon\phi + \varepsilon\theta\,\varepsilon\psi) \end{aligned} \quad (21)$$

Equation (20) has two solutions:

$$\Delta\phi = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (22)$$

$$\Delta\phi = \frac{-b-\sqrt{b^2-4ac}}{2a} \tag{23}$$

Equations (3, 17, 19, and 23) show that Euler angles can be corrected by removing attitude errors as follows:

$$\begin{aligned} \psi &= \hat{\psi} - \Delta\psi \\ \theta &= \hat{\theta} - \Delta\theta \\ \phi &= \hat{\phi} - \Delta\phi \end{aligned} \tag{24}$$

Using equation (24) for attitude correction leads to preserving the normality and orthogonality characteristics of the matrix of transformation. It should be noted that the derived formulas are used for attitude resetting, not for attitude propagation which may be done by the Rodrigues formulas, quaternions or etc.

## 4- Simulation

In order to investigate the performance of the proposed attitude correction algorithms, some simulations are performed. As a result, the dominant performance of the introduced algorithm in equation (20) over the other algorithm based on equation (5) is proved.

In the simulation scenario, 50 uniformly distributed random groups of Euler angle errors ($\Delta\varphi, \Delta\theta, \Delta\psi$) and random Euler angles ($(\varphi, \theta, \psi)$) are considered. These samples and their range are reported in Table 1 and equation (25), respectively. In the next step, small angles of rotation ($(\varepsilon\varphi, \varepsilon\theta, \varepsilon\psi)$) are determined using equation (12). It may also be noted that an estimation filter calculates these small rotation angles in real applications.

$$\begin{cases} \phi \in [-180 \quad 180]^\circ \\ \theta \in [-90 \quad 90]^\circ \\ \psi \in [-180 \quad 180]^\circ \end{cases}, \quad \begin{cases} \hat{\phi} = \phi + \Delta\phi \\ \hat{\theta} = \theta + \Delta\theta \\ \hat{\psi} = \psi + \Delta\psi \end{cases}, \quad \begin{cases} \Delta\phi \in [-30 \quad 30]^\circ \\ \Delta\theta \in [-30 \quad 30]^\circ \\ \Delta\psi \in [-30 \quad 30]^\circ \end{cases} \tag{25}$$

In Table 2, the results of three attitude correction methods are reported. In the first method, which is the typical attitude correction [15-18], the matrix of transformation $[\mathrm{T}]^{\mathrm{BN}}$ is determined employing equation (5). According to appendix C, normality and orthogonality conditions are applied to $[\mathrm{T}]^{\mathrm{BN}}$, then Euler angles are corrected using equation (6). Finally, the conventional method error would be:

$$\begin{aligned} &[\mathrm{T}]^{\mathrm{BN}} \equiv [\mathrm{t}_{i,j}] \\ &\hat{\Delta}\theta = \arcsin(-\mathrm{t}_{13}) - \theta \\ &\hat{\Delta}\phi = \arccos(\mathrm{t}_{33}/\cos\theta)\,\mathrm{sign}(\mathrm{t}_{23}) - \phi \\ &\hat{\Delta}\psi = \arccos(\mathrm{t}_{11}/\cos\theta)\,\mathrm{sign}(\mathrm{t}_{12}) - \psi \end{aligned} \tag{26}$$

The second method is based on the first-order approximation of rotation angles error, and Columns 5 to 7 of Table 2 belong to this algorithm. Using this algorithm, computing $\Delta\varphi, \Delta\theta, \Delta\psi$ by equation (13) leads to the correction of attitude angles through equation (24). Finally, first-order method error would be:

$$\begin{aligned} \hat{\Delta}\phi &= (\hat{\phi} - \Delta\phi) - \phi \\ \hat{\Delta}\theta &= (\hat{\theta} - \Delta\theta) - \theta \\ \hat{\Delta}\psi &= (\hat{\psi} - \Delta\psi) - \psi \end{aligned} \tag{27}$$

The third method is based on the second-order approximation of rotation angles error, introduced in equations (17), (19), and (23). Columns 8 to 9 of Table 2 report the results of this algorithm. This table reveals that attitude correction using the lowest error among other discussed methods belong to the second-order approximation technique. Equation (27) calculates the method error.
In order to compare the performance of the algorithms, some error definitions are used in Table 3. These errors are defined as:

$$e_1 = \left\| \hat{\Delta}\phi \quad \hat{\Delta}\theta \quad \hat{\Delta}\psi \right\|_2 \tag{28}$$

$$e_2 = \left\| \mathrm{I} - [\overline{\mathrm{T}}]^{\mathrm{B}\hat{\mathrm{N}}} [\mathrm{T}]^{\mathrm{B}\hat{\mathrm{N}}} \right\|_{\mathrm{Frobenius}} \tag{29}$$

$$e_3 = \left| 1 - [\overline{\mathrm{T}}]_1^{\mathrm{B}\hat{\mathrm{N}}} [\mathrm{T}]_1^{\mathrm{B}\hat{\mathrm{N}}} \right| + \left| 1 - [\overline{\mathrm{T}}]_2^{\mathrm{B}\hat{\mathrm{N}}} [\mathrm{T}]_2^{\mathrm{B}\hat{\mathrm{N}}} \right| + \left| 1 - [\overline{\mathrm{T}}]_3^{\mathrm{B}\hat{\mathrm{N}}} [\mathrm{T}]_3^{\mathrm{B}\hat{\mathrm{N}}} \right| \tag{30}$$

where $e_1$ represents error magnitude in corrected attitude angles, $e_2$ is orthogonality index, and $e_3$ evaluates normality characteristics. The term $[\mathrm{T}]_i^{\mathrm{B}\hat{\mathrm{N}}}$ $(i = 1,2,3)$ represents the $i^{th}$ column of the corrected matrix of transformation $[\mathrm{T}]^{\mathrm{B}\hat{\mathrm{N}}}$ in equation (30).

Table 3, is constructed by the data given in Table 1 and Table 2. Attitude correction errors of the algorithms ($e_1$) are illustrated in the following figures:

Figures (1) and (2) reveal that the least numerical error belongs to the method with second-order approximated formula. Additionally, based on Table 3, the first-order and the second-order approximation formulas do not induce normality and orthogonality errors (i.e. $\mathrm{e}_2$ and $\mathrm{e}_3$) to the matrix of transformation.

**Table 1. Random scenarios Description**

| Scenario Number | true Euler angles (in deg) | | | computed Euler angles (in deg) | | | small angles of rotation (in deg) | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\phi$ | $\theta$ | $\psi$ | $\hat{\phi}$ | $\hat{\theta}$ | $\hat{\psi}$ | $\varepsilon\phi$ | $\varepsilon\theta$ | $\varepsilon\psi$ |
| 1 | -146.040 | -14.061 | 29.510 | -150.470 | -14.238 | 19.991 | 3.630 | 1.597 | 10.463 |
| 2 | -117.820 | -37.852 | -57.367 | -123.740 | -33.899 | -59.382 | -0.547 | -6.100 | 5.422 |
| 3 | -23.880 | -10.134 | 59.545 | -21.552 | -16.270 | 53.050 | -6.419 | 1.842 | 6.017 |
| 4 | 159.230 | 10.623 | -82.694 | 168.420 | 14.207 | -78.405 | -4.811 | 8.124 | -1.932 |
| 5 | 160.380 | 19.175 | 98.570 | 170.120 | 14.550 | 89.763 | -2.913 | -9.129 | 11.136 |
| 6 | 64.762 | -4.387 | 97.450 | 59.027 | -6.784 | 95.789 | -3.135 | 5.441 | 2.356 |
| 7 | -9.909 | -83.719 | -41.908 | -8.800 | -74.202 | -45.241 | -6.550 | -6.585 | 1.830 |
| 8 | -13.597 | -16.365 | 123.980 | -13.119 | -6.541 | 130.500 | 8.372 | 6.000 | -6.951 |
| 9 | -8.386 | -21.436 | 52.392 | -15.252 | -31.139 | 60.630 | -4.174 | 10.195 | -4.697 |
| 10 | -157.630 | 7.653 | -10.179 | -165.440 | 15.445 | -7.018 | 5.999 | -8.514 | -4.189 |
| 11 | 173.720 | -39.119 | -22.277 | 166.370 | -44.044 | -20.102 | 6.820 | 2.427 | 2.541 |
| 12 | 120.650 | 58.349 | 16.112 | 121.540 | 52.423 | 22.779 | -2.161 | 5.276 | -5.793 |
| 13 | -57.638 | 39.553 | -15.124 | -49.754 | 40.093 | -17.100 | -6.014 | 1.523 | 7.053 |
| 14 | -79.195 | -76.653 | 99.653 | -80.796 | -69.359 | 97.241 | 7.069 | 1.284 | 3.832 |
| 15 | -7.373 | 17.383 | -93.124 | -1.016 | 26.077 | -85.153 | -8.432 | 5.314 | -5.111 |
| 16 | -47.914 | 1.324 | 117.460 | -52.124 | 2.020 | 111.990 | -1.317 | 4.158 | 5.370 |
| 17 | -154.530 | -73.875 | -154.520 | -155.670 | -81.408 | -156.330 | 3.078 | -7.003 | 3.082 |
| 18 | -151.580 | -63.664 | -163.990 | -148.210 | -64.533 | -159.180 | 1.640 | -0.245 | -7.806 |
| 19 | 46.072 | 12.702 | 71.330 | 53.580 | 15.949 | 78.885 | 0.869 | -7.767 | -5.571 |
| 20 | -154.610 | -63.842 | -11.629 | -151.180 | -67.682 | -7.344 | -0.492 | 3.957 | -7.352 |
| 21 | 77.008 | 5.527 | 54.567 | 76.965 | 5.287 | 54.663 | -0.171 | 0.174 | -0.100 |
| 22 | -137.060 | -19.708 | 156.080 | -130.280 | -16.010 | 150.890 | 7.394 | 0.219 | 3.066 |
| 23 | 93.203 | -50.700 | 168.310 | 93.404 | -52.641 | 169.990 | -0.274 | -1.933 | -1.843 |
| 24 | 87.247 | 26.944 | -1.559 | 93.766 | 18.736 | -5.520 | -5.940 | 8.805 | 6.845 |
| 25 | -114.790 | 49.004 | -39.528 | -107.260 | 57.107 | -47.341 | -8.413 | -1.668 | 13.410 |
| 26 | -165.800 | 64.837 | 178.930 | -165.580 | 74.790 | 179.820 | 0.243 | 9.902 | -0.697 |
| 27 | 174.540 | -76.501 | 133.720 | 173.170 | -75.215 | 142.170 | 0.684 | 1.214 | -7.112 |
| 28 | -174.110 | 39.289 | -58.052 | -173.180 | 47.742 | -60.570 | -7.481 | -3.510 | 2.894 |
| 29 | -68.388 | -0.616 | -9.349 | -68.117 | 9.024 | -0.332 | -1.823 | -9.593 | -8.987 |
| 30 | -26.477 | -7.860 | 176.350 | -19.822 | -13.995 | 170.610 | 6.037 | -7.023 | 4.784 |
| 31 | 117.810 | 5.295 | 81.136 | 112.080 | 6.334 | 81.374 | 1.930 | 5.485 | -0.907 |
| 32 | 19.800 | 23.124 | -33.552 | 9.820 | 16.508 | -41.000 | 11.402 | -0.838 | 4.052 |
| 33 | -121.770 | -2.050 | -29.298 | -129.370 | -11.691 | -25.961 | 11.181 | 5.218 | -2.882 |
| 34 | -167.930 | 84.702 | 161.860 | -173.080 | 91.956 | 161.740 | 2.417 | 6.999 | -5.130 |
| 35 | 46.172 | 55.819 | 119.670 | 54.260 | 53.414 | 109.710 | 0.533 | -5.161 | 16.280 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 36 | 47.240 | -21.569 | 64.566 | 56.874 | -20.141 | 59.431 | -2.829 | -8.536 | 1.464 |
| 37 | 65.320 | -38.547 | 125.190 | 57.000 | -28.808 | 122.350 | 3.466 | 10.785 | 6.962 |
| 38 | -148.780 | 55.420 | -89.390 | -144.000 | 61.432 | -88.765 | -6.104 | 2.580 | 3.564 |
| 39 | -53.564 | -11.055 | -128.280 | -60.648 | -9.307 | -128.710 | -5.627 | -4.274 | 1.460 |
| 40 | -120.670 | 24.955 | 145.110 | -115.010 | 33.587 | 146.430 | 8.838 | 4.214 | 1.110 |
| 41 | -85.806 | -6.013 | 66.476 | -89.420 | -11.038 | 67.861 | -3.197 | 5.210 | -0.698 |
| 42 | 105.460 | 68.569 | 147.070 | 105.810 | 62.146 | 155.580 | -3.347 | -5.890 | -8.290 |
| 43 | 58.701 | 73.546 | 45.468 | 65.053 | 78.347 | 43.252 | 2.568 | -4.651 | 8.427 |
| 44 | 151.200 | -73.871 | 35.923 | 143.580 | -73.330 | 26.995 | 2.001 | 0.357 | 16.007 |
| 45 | -91.614 | 55.684 | -42.909 | -94.051 | 59.937 | -35.220 | -2.014 | -4.243 | -9.751 |
| 46 | 95.673 | -43.992 | -89.887 | 98.079 | -40.286 | -98.890 | -3.680 | 2.285 | 7.422 |
| 47 | -41.559 | 84.952 | 88.313 | -35.437 | 92.501 | 83.517 | 7.518 | -1.381 | 10.860 |
| 48 | 23.008 | -72.809 | -13.919 | 20.109 | -72.175 | -20.171 | 0.714 | -0.910 | 8.984 |
| 49 | 5.647 | 40.360 | -66.308 | 9.209 | 34.450 | -60.495 | 4.169 | 5.290 | -3.813 |
| 50 | 63.928 | 53.609 | -160.210 | 72.096 | 45.440 | -151.290 | 8.020 | -5.036 | -2.292 |

**Table 2. Error comparison of three different attitude correction algorithms**

| Sc. | Conventional algorithm | | | 1st order approximation algorithm | | | 2nd order approximation algorithm | | |
|---|---|---|---|---|---|---|---|---|---|
| No. | $\lvert\hat{\Delta}\phi^{\circ}\rvert$ | $\lvert\hat{\Delta}\theta^{\circ}\rvert$ | $\lvert\hat{\Delta}\psi^{\circ}\rvert$ | $\lvert\hat{\Delta}\phi^{\circ}\rvert$ | $\lvert\hat{\Delta}\theta^{\circ}\rvert$ | $\lvert\hat{\Delta}\psi^{\circ}\rvert$ | $\lvert\hat{\Delta}\phi^{\circ}\rvert$ | $\lvert\hat{\Delta}\theta^{\circ}\rvert$ | $\lvert\hat{\Delta}\psi^{\circ}\rvert$ |
| 1 | 0.394 | 0.249 | 0.139 | 0.352 | 0.083 | 10.524 | 0.098 | 0.008 | 0.112 |
| 2 | 0.081 | 0.316 | 0.141 | 0.063 | 0.375 | 5.355 | 0.146 | 0.108 | 0.067 |
| 3 | 0.251 | 0.175 | 0.043 | 0.157 | 0.101 | 5.799 | 0.026 | 0.042 | 0.081 |
| 4 | 0.198 | 0.224 | 0.430 | 0.019 | 0.504 | 2.030 | 0.010 | 0.176 | 0.123 |
| 5 | 0.561 | 1.135 | 0.458 | 0.289 | 1.751 | 11.180 | 0.078 | 0.026 | 0.097 |
| 6 | 0.036 | 0.073 | 0.129 | 0.035 | 0.173 | 2.343 | 0.009 | 0.027 | 0.021 |
| 7 | 4.897 | 0.444 | 2.039 | 1.343 | 0.229 | 3.557 | 0.393 | 0.083 | 0.798 |
| 8 | 0.090 | 0.299 | 0.312 | 0.402 | 0.439 | 6.627 | 0.065 | 0.114 | 0.349 |
| 9 | 0.272 | 0.721 | 0.539 | 1.123 | 1.065 | 4.107 | 0.165 | 0.135 | 0.306 |
| 10 | 0.504 | 0.331 | 0.376 | 0.551 | 0.076 | 5.093 | 0.224 | 0.148 | 0.030 |
| 11 | 0.150 | 0.207 | 0.158 | 0.401 | 0.302 | 3.212 | 0.180 | 0.202 | 0.124 |
| 12 | 0.428 | 0.277 | 0.620 | 0.971 | 0.225 | 6.732 | 0.061 | 0.020 | 0.174 |
| 13 | 0.142 | 0.330 | 0.138 | 0.215 | 0.227 | 7.192 | 0.156 | 0.233 | 0.071 |
| 14 | 1.027 | 0.189 | 0.776 | 0.515 | 0.119 | 3.429 | 0.008 | 0.040 | 0.072 |
| 15 | 0.305 | 0.462 | 0.727 | 0.331 | 0.742 | 5.031 | 0.054 | 0.325 | 0.095 |
| 16 | 0.095 | 0.154 | 0.009 | 0.141 | 0.361 | 5.314 | 0.010 | 0.010 | 0.043 |
| 17 | 0.211 | 0.090 | 0.792 | 1.186 | 0.117 | 1.758 | 0.057 | 0.032 | 0.115 |
| 18 | 0.001 | 0.123 | 0.094 | 0.002 | 0.057 | 7.847 | 0.033 | 0.034 | 0.053 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 19 | 0.068 | 0.434 | 0.159 | 0.244 | 0.898 | 5.425 | 0.070 | 0.040 | 0.044 |
| 20 | 0.648 | 0.048 | 0.542 | 0.806 | 0.022 | 6.706 | 0.056 | 0.042 | 0.035 |
| 21 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.100 | 0.000 | 0.000 | 0.000 |
| 22 | 0.004 | 0.221 | 0.076 | 0.176 | 0.091 | 3.369 | 0.132 | 0.047 | 0.028 |
| 23 | 0.047 | 0.008 | 0.040 | 0.092 | 0.010 | 1.769 | 0.000 | 0.001 | 0.005 |
| 24 | 0.355 | 0.088 | 0.906 | 0.618 | 0.015 | 6.254 | 0.157 | 0.085 | 0.024 |
| 25 | 0.533 | 1.222 | 1.217 | 0.712 | 0.786 | 14.731 | 0.219 | 0.090 | 0.637 |
| 26 | 0.287 | 0.105 | 2.625 | 0.589 | 0.050 | 0.113 | 0.006 | 0.050 | 0.000 |
| 27 | 0.168 | 0.066 | 0.173 | 0.570 | 0.093 | 7.680 | 0.040 | 0.004 | 0.026 |
| 28 | 0.037 | 0.282 | 0.382 | 0.010 | 0.214 | 3.200 | 0.019 | 0.050 | 0.277 |
| 29 | 0.709 | 0.105 | 0.274 | 1.518 | 0.037 | 8.736 | 0.012 | 0.046 | 0.032 |
| 30 | 0.198 | 0.121 | 0.018 | 0.664 | 0.191 | 3.966 | 0.131 | 0.091 | 0.024 |
| 31 | 0.011 | 0.027 | 0.089 | 0.013 | 0.046 | 0.872 | 0.002 | 0.026 | 0.040 |
| 32 | 0.162 | 0.474 | 0.105 | 0.431 | 0.232 | 4.735 | 0.378 | 0.002 | 0.063 |
| 33 | 0.208 | 0.057 | 0.433 | 0.327 | 0.055 | 1.729 | 0.097 | 0.135 | 0.080 |
| 34 | 2.037 | 0.181 | 5.284 | 2.163 | 0.149 | 2.856 | 0.100 | 0.043 | 0.008 |
| 35 | 0.006 | 0.791 | 0.810 | 0.365 | 1.166 | 16.743 | 0.349 | 0.516 | 0.469 |
| 36 | 0.241 | 0.090 | 0.313 | 0.274 | 0.477 | 1.911 | 0.208 | 0.162 | 0.229 |
| 37 | 0.014 | 1.055 | 0.604 | 0.038 | 1.040 | 6.829 | 0.203 | 0.305 | 0.335 |
| 38 | 0.269 | 0.028 | 0.145 | 0.893 | 0.034 | 4.354 | 0.009 | 0.169 | 0.003 |
| 39 | 0.164 | 0.024 | 0.201 | 0.138 | 0.031 | 1.553 | 0.100 | 0.026 | 0.181 |
| 40 | 0.095 | 0.230 | 0.342 | 0.380 | 0.234 | 2.024 | 0.052 | 0.102 | 0.241 |
| 41 | 0.021 | 0.078 | 0.156 | 0.075 | 0.100 | 0.679 | 0.025 | 0.007 | 0.044 |
| 42 | 0.644 | 0.239 | 0.509 | 1.663 | 0.323 | 9.676 | 0.100 | 0.030 | 0.192 |
| 43 | 0.558 | 0.282 | 0.590 | 0.165 | 0.348 | 8.600 | 0.283 | 0.033 | 0.320 |
| 44 | 1.152 | 0.193 | 0.438 | 0.843 | 0.050 | 15.423 | 0.579 | 0.053 | 0.315 |
| 45 | 0.354 | 0.255 | 0.253 | 0.836 | 0.374 | 9.074 | 0.031 | 0.019 | 0.039 |
| 46 | 0.281 | 0.100 | 0.167 | 0.191 | 0.283 | 7.571 | 0.013 | 0.081 | 0.014 |
| 47 | 10.077 | 0.223 | 7.228 | 18.114 | 0.076 | 7.185 | 0.042 | 0.047 | 0.094 |
| 48 | 0.237 | 0.019 | 0.361 | 0.314 | 0.026 | 9.312 | 0.022 | 0.018 | 0.058 |
| 49 | 0.057 | 0.140 | 0.178 | 0.469 | 0.323 | 4.063 | 0.061 | 0.080 | 0.123 |
| 50 | 0.278 | 0.339 | 0.752 | 1.591 | 0.100 | 4.234 | 0.533 | 0.078 | 0.594 |
| average | 0.591 | 0.267 | 0.666 | 0.868 | 0.296 | 5.572 | 0.116 | 0.085 | 0.146 |

**Table 3. Performance comparison of three different Euler angles correction methods**

| Sc. No. | Conventional method $e_1^\circ$ | Conventional method $e_2$ | Conventional method $e_3$ | 1st order approximation method $e_1^\circ$ | 1st order approximation method $e_2$ | 1st order approximation method $e_3$ | 2nd order approximation method $e_1^\circ$ | 2nd order approximation method $e_2$ | 2nd order approximation method $e_3$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.486 | 0.002 | 0.001 | 10.530 | 0.000 | 0.000 | 0.149 | 0.000 | 0.000 |
| 2 | 0.356 | 0.000 | 0.000 | 5.369 | 0.000 | 0.000 | 0.193 | 0.000 | 0.000 |
| 3 | 0.309 | 0.001 | 0.000 | 5.802 | 0.000 | 0.000 | 0.095 | 0.000 | 0.000 |
| 4 | 0.524 | 0.001 | 0.001 | 2.092 | 0.000 | 0.000 | 0.216 | 0.000 | 0.000 |
| 5 | 1.347 | 0.004 | 0.003 | 11.320 | 0.000 | 0.000 | 0.128 | 0.000 | 0.000 |
| 6 | 0.153 | 0.000 | 0.000 | 2.350 | 0.000 | 0.000 | 0.035 | 0.000 | 0.000 |
| 7 | 5.323 | 0.001 | 0.001 | 3.809 | 0.000 | 0.000 | 0.893 | 0.000 | 0.000 |
| 8 | 0.441 | 0.002 | 0.002 | 6.653 | 0.000 | 0.000 | 0.373 | 0.000 | 0.000 |
| 9 | 0.941 | 0.002 | 0.001 | 4.389 | 0.000 | 0.000 | 0.373 | 0.000 | 0.000 |
| 10 | 0.710 | 0.002 | 0.001 | 5.123 | 0.000 | 0.000 | 0.270 | 0.000 | 0.000 |
| 11 | 0.300 | 0.000 | 0.000 | 3.251 | 0.000 | 0.000 | 0.298 | 0.000 | 0.000 |
| 12 | 0.803 | 0.000 | 0.000 | 6.806 | 0.000 | 0.000 | 0.186 | 0.000 | 0.000 |
| 13 | 0.385 | 0.001 | 0.001 | 7.199 | 0.000 | 0.000 | 0.289 | 0.000 | 0.000 |
| 14 | 1.301 | 0.000 | 0.000 | 3.469 | 0.000 | 0.000 | 0.083 | 0.000 | 0.000 |
| 15 | 0.914 | 0.002 | 0.001 | 5.096 | 0.000 | 0.000 | 0.343 | 0.000 | 0.000 |
| 16 | 0.181 | 0.000 | 0.000 | 5.328 | 0.000 | 0.000 | 0.045 | 0.000 | 0.000 |
| 17 | 0.825 | 0.000 | 0.000 | 2.124 | 0.000 | 0.000 | 0.132 | 0.000 | 0.000 |
| 18 | 0.155 | 0.000 | 0.000 | 7.847 | 0.000 | 0.000 | 0.071 | 0.000 | 0.000 |
| 19 | 0.468 | 0.001 | 0.001 | 5.504 | 0.000 | 0.000 | 0.092 | 0.000 | 0.000 |
| 20 | 0.846 | 0.000 | 0.000 | 6.754 | 0.000 | 0.000 | 0.078 | 0.000 | 0.000 |
| 21 | 0.000 | 0.000 | 0.000 | 0.100 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 22 | 0.234 | 0.000 | 0.000 | 3.375 | 0.000 | 0.000 | 0.143 | 0.000 | 0.000 |
| 23 | 0.062 | 0.000 | 0.000 | 1.771 | 0.000 | 0.000 | 0.005 | 0.000 | 0.000 |
| 24 | 0.977 | 0.002 | 0.002 | 6.284 | 0.000 | 0.000 | 0.180 | 0.000 | 0.000 |
| 25 | 1.804 | 0.006 | 0.004 | 14.769 | 0.000 | 0.000 | 0.680 | 0.000 | 0.000 |
| 26 | 2.643 | 0.001 | 0.001 | 0.601 | 0.000 | 0.000 | 0.051 | 0.000 | 0.000 |
| 27 | 0.250 | 0.000 | 0.000 | 7.702 | 0.000 | 0.000 | 0.048 | 0.000 | 0.000 |
| 28 | 0.476 | 0.001 | 0.000 | 3.207 | 0.000 | 0.000 | 0.282 | 0.000 | 0.000 |
| 29 | 0.767 | 0.003 | 0.002 | 8.867 | 0.000 | 0.000 | 0.057 | 0.000 | 0.000 |
| 30 | 0.233 | 0.001 | 0.001 | 4.026 | 0.000 | 0.000 | 0.161 | 0.000 | 0.000 |
| 31 | 0.093 | 0.000 | 0.000 | 0.873 | 0.000 | 0.000 | 0.048 | 0.000 | 0.000 |
| 32 | 0.512 | 0.002 | 0.001 | 4.761 | 0.000 | 0.000 | 0.383 | 0.000 | 0.000 |
| 33 | 0.483 | 0.002 | 0.002 | 1.761 | 0.000 | 0.000 | 0.185 | 0.000 | 0.000 |
| 34 | 5.666 | 0.001 | 0.000 | 3.585 | 0.000 | 0.000 | 0.109 | 0.000 | 0.000 |
| 35 | 1.132 | 0.008 | 0.006 | 16.788 | 0.000 | 0.000 | 0.780 | 0.000 | 0.000 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 36 | 0.405 | 0.001 | 0.000 | 1.989 | 0.000 | 0.000 | 0.349 | 0.000 | 0.000 |
| 37 | 1.215 | 0.003 | 0.002 | 6.908 | 0.000 | 0.000 | 0.496 | 0.000 | 0.000 |
| 38 | 0.307 | 0.000 | 0.000 | 4.444 | 0.000 | 0.000 | 0.170 | 0.000 | 0.000 |
| 39 | 0.261 | 0.000 | 0.000 | 1.560 | 0.000 | 0.000 | 0.208 | 0.000 | 0.000 |
| 40 | 0.423 | 0.001 | 0.001 | 2.073 | 0.000 | 0.000 | 0.267 | 0.000 | 0.000 |
| 41 | 0.175 | 0.000 | 0.000 | 0.690 | 0.000 | 0.000 | 0.051 | 0.000 | 0.000 |
| 42 | 0.855 | 0.001 | 0.001 | 9.823 | 0.000 | 0.000 | 0.218 | 0.000 | 0.000 |
| 43 | 0.859 | 0.001 | 0.001 | 8.608 | 0.000 | 0.000 | 0.429 | 0.000 | 0.000 |
| 44 | 1.248 | 0.006 | 0.005 | 15.446 | 0.000 | 0.000 | 0.661 | 0.000 | 0.000 |
| 45 | 0.504 | 0.001 | 0.001 | 9.120 | 0.000 | 0.000 | 0.054 | 0.000 | 0.000 |
| 46 | 0.342 | 0.001 | 0.000 | 7.579 | 0.000 | 0.000 | 0.084 | 0.000 | 0.000 |
| 47 | 12.404 | 0.003 | 0.002 | 19.487 | 0.000 | 0.000 | 0.113 | 0.000 | 0.000 |
| 48 | 0.432 | 0.001 | 0.000 | 9.317 | 0.000 | 0.000 | 0.064 | 0.000 | 0.000 |
| 49 | 0.233 | 0.000 | 0.000 | 4.103 | 0.000 | 0.000 | 0.159 | 0.000 | 0.000 |
| 50 | 0.870 | 0.001 | 0.001 | 4.524 | 0.000 | 0.000 | 0.802 | 0.000 | 0.000 |
| average | 1.053 | 0.001 | 0.001 | 5.899 | 0.000 | 0.000 | 0.232 | 0.000 | 0.000 |
| Maximum | 12.404 | 0.008 | 0.006 | 19.487 | 0.000 | 0.000 | 0.893 | 0.000 | 0.000 |

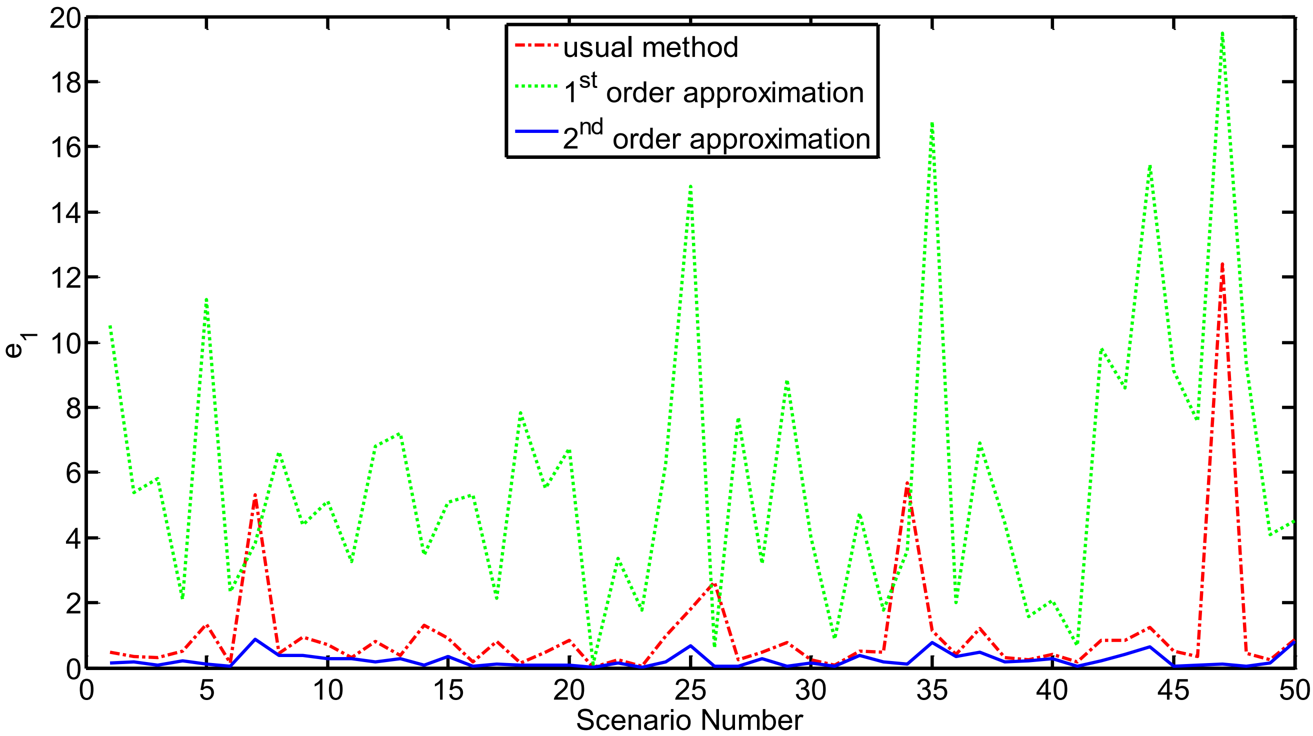


**Fig. 1. Error amplitude Comparison in three different attitude resetting methods**

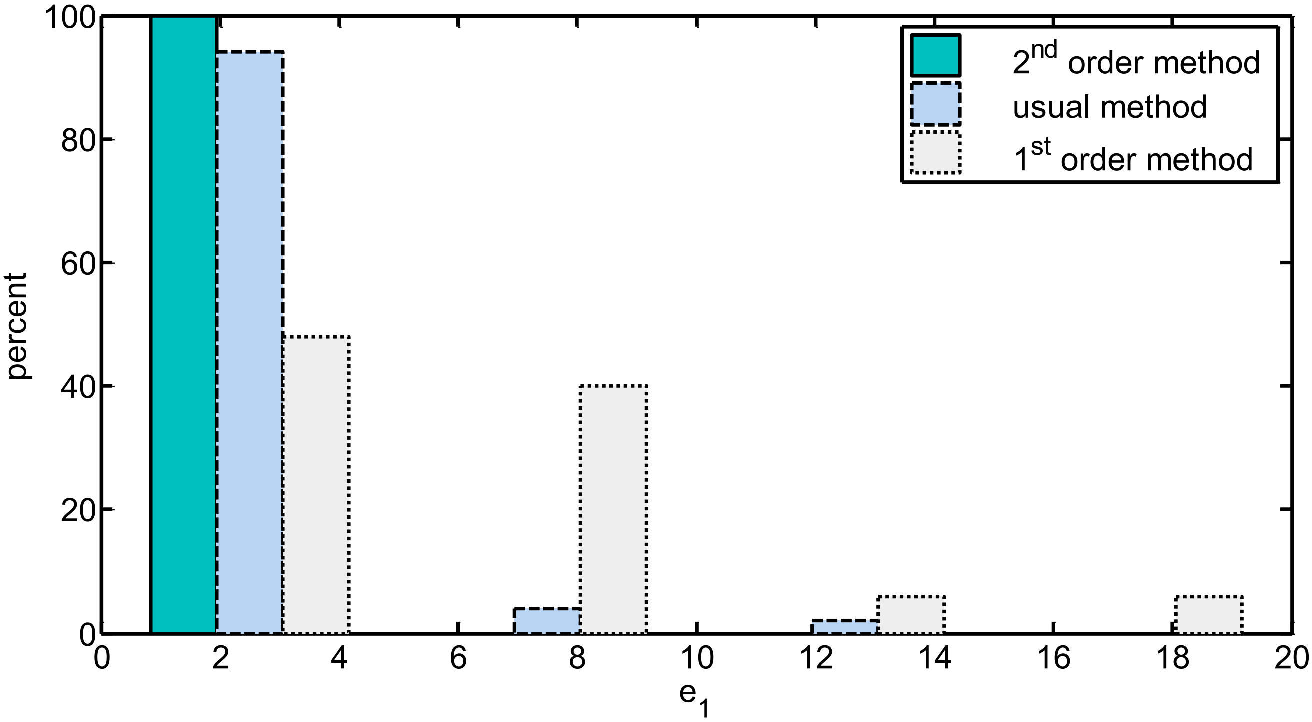


**Fig. 2. Error histogram in three different attitude resetting methods**

## 5- Conclusion

Euler angles error was obtained in terms of small rotation angles and vice versa. These equations revealed that small angles of rotation and Euler angles error relation are nonlinear. Thus, a first-order approximation was made to remove the nonlinearity relation between small angles of rotation and Euler angles error. An analytical attitude correction algorithm was found on this approximation. Numerical simulations proved that the proposed method for attitude correction is more accurate than the conventional method. Another advantage of the proposed method appears when a significant error of attitude occurs. Unlike the conventional method, the new algorithm preserves normality and orthogonality characteristics of the matrix of transformation in this situation.

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