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Original Article

On two generation methods for the simple linear group $PSL(3,7)$

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ABSTRACT: A finite group G is said to be (l,m,n)-*generated*, if it is a quotient group of the triangle group $T(l,m,n) = \langle x,y,z|x^l = y^m = z^n = xyz = 1\rangle$. In [J. Moori, (p,q,r)-*generations for the Janko groups* J_1 *and* J_2, Nova J. Algebra and Geometry, **2** (1993), no. 3, 277–285], Moori posed the question of finding all the (p,q,r) triples, where p, q and r are prime numbers, such that a non-abelian finite simple group G is (p,q,r)-generated. Also for a finite simple group G and a conjugacy class X of G, the *rank* of X in G is defined to be the minimal number of elements of X generating G. In this paper we investigate these two generational problems for the group $PSL(3,7)$, where we will determine the (p,q,r)-generations and the ranks of the classes of $PSL(3,7)$. We approach these kind of generations using the structure constant method. GAP [The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.9.3;* 2018. (http://www.gap-system.org)] is used in our computations.

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(Dedicated to Professor Jamshid Moori)

1. Introduction

The problem of generation of finite groups has great interest and has many applications to groups and their representations. The classification of finite simple groups is involved heavily and play a pivotal role in most general results on the generation of finite groups. The study of generating sets in finite groups has a rich history, with numerous applications. We are interested in two kinds of generations of a finite simple group G, namely the (p,q,r)-generation and the ranks of conjugacy classes of G.

A finite group G is said to be (l,m,n)-*generated*, if $G = \langle x,y\rangle$, with $o(x) = l$, $o(y) = m$ and $o(z) = n$, where $z = (xy)^{-1}$. Here $[x] = lX$ is the conjugacy class of x in G and the elements in this class are of order l. Similarly for the classes $[y] = mY$ and $[z] = nZ$. In this case G is also a quotient group of the triangular group $T(l,m,n)$ and, by definition of the triangular group, G is also $(\sigma(l),\sigma(m),\sigma(n))$-generated group for any $\sigma \in S_3$. Therefore we may assume that $l \leq m \leq n$. In a series of papers [17, 16, 18, 19, 20, 23, 24], Moori and Ganief established all possible (p,q,r)-generations, p, q and r are distinct primes, of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2 and F_{22}. Ashrafi in [2, 3] did the same for the sporadic groups He and HN. Also Darafsheh and Ashrafi established in [13, 12, 14, 15], the (p,q,r)-generations of the sporadic groups Co_1, Ru, $O^{'}N$ and Ly. Basheer and Seretlo in [4] and [9] established the (p,q,r)-generations of the Mathieu sporadic group M_{22} and the alternating group

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A_{10} respectively. They also [8] looked at the 2 generation method where besides the (p, q, r) generations they also established the conjugacy class ranks of the group $PSL(3, 7)$.

From another side, for a finite simple group G and non-trivial class nX of G, the *rank* of nX in G, denoted by $rank(G : nX)$, is defined to be the minimal number of elements of nX generating G. One of the applications of ranks of conjugacy classes of a finite group is that they are involved in the computations of the covering number of the finite simple group. We recall from Zisser [27] that for a finite simple group G, the *covering number* of G is the smallest integer n such that $C^n = G$, for all non-trivial conjugacy classes C of G and by C^n we mean $\{c_1c_2 \cdots c_n | \ c_1, c_2, \cdots, c_n \in C\}$. In [22, 25, 26], J. Moori computed the ranks of the involutory classes of the Fischer sporadic simple group Fi_{22}. He found that $rank(Fi_{22}{:}2B) = rank(Fi_{22}{:}2C) = 3$, while $rank(Fi_{22}{:}2A) \in \{5, 6\}$. The work of Hall and Soicher [21] implies that $rank(Fi_{22}{:}2A) = 6$. Then in a considerable number of publications (see the list of references of [6]) various authors explored the ranks for many of the sporadic simple groups.

The motivation for studying the (p, q, r)-generations and the ranks of classes in a finite simple group G is outlined in the above mentioned papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

This paper intends to be a continuation to the above series on simple groups, where we will establish all the (p, q, r)-generations together with the ranks of the conjugacy classes of the projective special linear group $PSL(3, 7)$. Note that, in general, if G is a $(2, 2, n)$-generated group, then G is a dihedral group and therefore G is not simple. Also by [10], if G is a non-abelian (l, m, n)-generated group, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Thus for our purpose of establishing the (p, q, r)-generations of $G = PSL(3, 7)$, the only cases we need to consider are when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Therefore excluding the triples $(2, 2, p)$ and those that do not satisfy the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, we remain with 380 triples (p, q, r), $p \le q \le r$ to consider. We found that out of these 380 triples, 348 of them generate $PSL(3, 7)$. The main result on the (p, q, r)-generations of the projective special linear group $PSL(3, 7)$ can be summarized in the following theorem.

Theorem 1.1. *Let $S = \{B, C, D\}$ and $T := \{A, B, C, D, E, F\}$. The projective special linear group $PSL(3, 7)$ is generated by all the triples (pX, qY, rZ), p, q and r are primes dividing $|PSL(3, 7)|$ if and only if (pX, qY, rZ) is one of the following triples:*

1. $(2A, 3A, 19Y)$;$(2A, 7X, 7X)$; $(2A, 7X, 19Y)$; $(2A, 19Y, 19Z)$, $X \in S$; Y, $Z \in T$;
2. $(3A, 3A, 7X)$; $(3A, 3A, 19Y)$;$(3A, 7X, 7X)$; $(3A, 7X, 19Y)$;$(3A, 19Y, 19Z)$; $X \in S$; Y, $Z \in T$;
3. $(7A, 7X, 7X)$; $(7V, 7W, 7X)$; $(7A, 7X, 19Y)$; $(7A, 19Y, 19Z)$;$(7X, 19Y, 19Z)$; V, W, $X \in S$;Y, $Z \in T$;
4. $(19X, 19Y 19Z)$; X, Y, $Z \in T$.

The proof of Theorem 1.1 will be done through sequence of propositions that will be established in Subsections 3.1, 3.2 and 3.3.

Also the main result on the ranks of non trivial classes of G can be summarized in the following theorem.

Theorem 1.2. *Let G be the projective special linear group $PSL(3, 7)$. Then*

1. $rank(G{:}2A) = rank(G{:}7A) = 3$,
2. $rank(G{:}nX) = 2$ *for all* $nX \notin \{1A, 2A, 7A\}$.

The proof of Theorem 1.2 will be established in Propositions 4.1, 4.2, 4.3 and 4.4.

In [5],the ranks of the classes of the group A_{10} using the structure constant method were determined. In this paper we use the same technique to determine the (p, q, r)-generations and ranks of conjugacy classes of $PSL(3, 7)$. Therefore for the notation, description of the structure constant method and known results, we follow precisely [7, 5, 6, 4].

2. The projective special linear group $PSL(3, 7)$

The projective special linear group $PSL(3, 7)$ is a simple group of order $1876896 = 2^5 \times 3^2 \times 7^3 \times 19$. By the Atlas [11], the group $PSL(3, 7)$ has exactly 22 conjugacy classes of its elements, of which 12 of these classes have elements of prime orders. These are the classes $2A$, $3A$, $7A$, $7B$, $7C$, $7D$, $19A$, $19B$, $19C$, $19D$, $19E$ and $19F$. Also $PSL(3, 7)$ has 8 conjugacy classes of maximal subgroups, where representatives of these classes of maximal subgroups can be taken as follows:

$H_1 \cong H_2 = 7^2{:}SL(2, 7){:}2$	$H_3 \cong H_4 \cong H_5 = PSL(3, 2){:}2$	$H_6 = 3.A_4{:}2$
$H_7 = 3^2{:}Q_8$	$H_8 = 19{:}3$	

Throughout this paper and unless otherwise stated, by G, we always mean the projective special linear group $PSL(3,7)$. For a subgroup H of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$, we let $h(g,H)$ be the number of conjugates of H in G containing g. This number is given by $\chi_H(g)$, where χ_H is the permutation character of G with action on the conjugates of H. Using Theorem 2.2 of [6] we computed the values of $h(g,H_i)$ for all the non-identity classes of elements and all the maximal subgroups $H_i,\ 1 \le i \le 8$, of G and we list these values in Table 1.

Table 1: The values $h(g,H_i),\ 1 \le i \le 8$ for non-identity classes and maximal subgroups of $PSL(3,7)$

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$2A$	9	9	98	98	98	196	84	0
$3A$	3	3	6	6	6	13	4	24
$4A$	1	1	2	2	2	4	12	0
$6A$	3	3	2	2	2	1	0	0
$7A$	8	8	0	0	0	0	0	0
$7B$	1	1	0	0	7	0	0	0
$7C$	1	1	7	0	0	0	0	0
$7D$	1	1	0	7	0	0	0	0
$8A$	1	1	2	2	2	0	0	0
$8B$	1	1	2	2	2	0	0	0
$14A$	2	2	0	0	0	0	0	0
$16A$	1	1	0	0	0	0	0	0
$16B$	1	1	0	0	0	0	0	0
$16C$	1	1	0	0	0	0	0	0
$16D$	1	1	0	0	0	0	0	0
$19A$	0	0	0	0	0	0	0	1
$19B$	0	0	0	0	0	0	0	1
$19C$	0	0	0	0	0	0	0	1
$19D$	0	0	0	0	0	0	0	1
$19E$	0	0	0	0	0	0	0	1
$19F$	0	0	0	0	0	0	0	1

3. The (p,q,r)-generations of the $PSL(3,7)$

In this section we investigate all the generation of $PSL(3,7) := G$ by the triples (pX, qY, rZ), p, q and r are primes that divide the order of G. That is $p, q, r \in \{2, 3, 7, 19\}$.

3.1. The $(2,q,r)$-generations of G

The $(2,q,r)$-generations comprise three cases, namely $(2,3,r)$-, $(2,7,r)$- and $(2,19,r)$-generations. The condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, implies that if G is $(2A, 3A, rZ)$-generated, then we must have $r > 6$. That is $r = 7$ or $r = 19$. Throughout the paper, let S and T be as in Theorem 1.1; that is $S = \{B, C, D\}$ and $T = \{A, B, C, D, E, F\}$.

Proposition 3.1. *G is $(2A, 3A, 19Y)$-generated group for $Y \in T$.*

Proof. The computations with GAP [1] show that $\Delta_G(2A, 3A, 19Y) = 57$, for all $Y \in T$. From Table 1 we can see that only $H_8 = 19{:}3$ is the maximal subgroup of G that contains elements of order 19. However we can see that the order of H_8 is an odd and thus there is no fusion from this subgroup into the class $2A$ of G. It follows that there is no contribution from any maximal subgroup of G to $\Delta^*_G(2A, 3A, 19X)$, for any $X \in T$. Thus $\Delta^*_G(2A, 3A, 19Y) = \Delta_G(2A, 3A, 19Y) = 57$, for all $Y \in T$. Hence G is generated by all the triples $(2A, 3A, 19Y)$, for $Y \in T$. □

Proposition 3.2. *G is neither $(2A, 7A, 7X)$- nor $(2A, 7A, 19Y)$-generated group for all $X \in \{A, B, C, D\}$ and $Y \in T$. Also G is not $(2A, 7W, 7X)$-generated group, for all $W,\ X \in S, X \neq W$.*

Proof. The GAP computations reveals that $\Delta_G(2A, 7A, 7X) = \Delta_G(2A, 7A, 19Y) = 0$ for all $X \in \{A, B, C, D\}$ and $Y \in T$. Also, the computations reveals that $\Delta_G(2A, 7W, 7X) = 0$ for all $W,\ X \in S,\ X \neq W$. Hence the result. □

Proposition 3.3. *The group G is $(2A, 7X, 7X)$-generated, $X \in S$*

Proof. From Table 1 there are five maximal subgroups of G whose elements fuse into either $7B, 7C$ or $7D$. These are subgroups $H_1 = H_2$ and $H_3 = H_4 = H_5$. The intersection of any two of these maximal subgroups are as follows: $H_1 \cap H_2 \cong SL(2,7){:}2$, $H_1 \cap H_3 \cong D_{16}$, $H_1 \cap H_4 \cong D_{12}$ and $H_1 \cap H_5 \cong 7{:}6$. Note that from $SL(2,7){:}2$ we have $7a \to 7A$.

Similar intersections for H_3, H_4 and H_5 hold for H_2. Also $H_1 \cap H_2 \cap H_5 \cong \mathbb{Z}_6$, which clearly has no element of order 7. Computations show that $h(7D, 7{:}6) = 7$ and also $\sum_{7:6}(2a, 7a, 7a) = 0$. Also $7a$ in H_3 fuses to $7B$, $7a$ in H_4 fuses into $7C$ and $7a$ in H_5 into $7D$. We also have from H_1, $7a \to 7A, 7b \to 7A$ while $7c \to 7B$, $7d \to 7C$, $7e \to 7D$. The fusions also hold for H_2. We also have that the computations show that $\Delta_G(2A, 7X, 7X) = 147$ for all $X \in S$. Also $\sum_{H_1}(2A, 7A, 7A) = \sum_{H_2}(2A, 7A, 7A) = 0$. We also have $\sum_{H_3}(2a, 7a, 7a) = \sum_{H_4}(2a, 7a, 7a) = \sum_{H_5}(2a, 7a, 7a) = 7$ From Table 1, we have $h(7X, H_1) = h(7X, H_2) = 1$ for each $X \in S$. We also have $7a$ in H_3 fuses into $7C$ and $7a$ in H_4 fuses into $7B$ and $7a$ in H_5 into $7D$ and computations give us and $h(7C, H_3) = h(7B, H_4) = h(7D, H_5) = 7$. Finally

$$\begin{aligned}
\Delta^*_G(2A, 7B, 7B) &= \Delta_G(2A, 7B, 7B) - 1 \times \sum_{H_1}(2a, 7c, 7c) - 1 \times \sum_{H_2}(2a, 7c, 7c) \\
&- 7 \times \sum_{H_3}(2a, 7a, 7a) \\
&= 147 - 13 - 13 - 49 \\
&= 72 > 0
\end{aligned}$$

Hence $(2A, 7B, 7B)$ generate G.
Similarly $\Delta^*_G(2A, 7C, 7C) = 72$. and hence $(2A, 7C, 7C)$ generates G.
Finally

$$\begin{aligned}
\Delta^*_G(2A, 7D, 7D) &= \Delta_G(2A, 7D, 7D) - 1 \times \sum_{H_1}(2a, 7e, 7e) - 1 \times \sum_{H_2}(2a, 7e, 7e) \\
&- 7 \times \sum_{H_5}(2a, 7a, 7a) + 7 \times \sum_{H_1 \cap H_5}(2a, 7a, 7a) \\
&= 147 - 13 - 13 - 49 + 0 \\
&= 72 > 0
\end{aligned}$$

Again, $(2A, 7D, 7D)$, generates G.
The Proposition is proved. □

Proposition 3.4. *G is $(2A, 7X, 19Y)$-generated group for all $X \in S, Y \in T$*

Proof. The computations with GAP show that $\Delta_G(2A, 7X, 19Y) = 57$, for all $X \in S, Y \in T$. From Table 1, we can see that only $H_8 = 19{:}3$ is the maximal subgroup of G that contains elements of order19. However we can see that the order of H_8 is neither divisible by 2 nor by 7 and thus there is no fusion from classes of H_8 into the classes $2A$ and $7X$ of G. It follows that there is no contribution from any maximal subgroup of G to $\Delta^*_G(2A, 7X, 19Y)$, for any $X \in T$. Thus $\Delta^*_G(2A, 7X, 19Y) = \Delta_G(2A, 7X, 19Y) = 57$, for all $X \in S, Y \in T$. Hence G is generated by all the triples $(2A, 7X, 19Y)$, for $X \in S, Y \in T$. □

We now look at the last case of the $(2, q, r)$-generations, namely the $(2, 19, 19)$-generations.

Proposition 3.5. *G is $(2A, 19Y, 19Z)$-generated group for all $Y,\ Z \in T$.*

Proof. The computations with GAP show that $\Delta_G(2A, 19Y, 19Z) = 171$, for all $Y,\ Z \in T$. The treatment is same as in Propositions 3.1 and 3.3 and thus $\Delta^*_G(2A, 19Y, 19Z) = \Delta_G(2A, 19Y, 19Z) = 171$, for all $Y,\ Z \in T$. Hence G is generated by all the triples $(2A, 19Y, 19Z)$, for $Y,\ Z \in T$. □

3.2. The $(3, q, r)$-generations of G

In this subsection we consider all the $(3, q, r)$-generations, which constitutes the cases $(3, 3, r)$-, $(3, 7, r)$- and $(3, 19, r)$-generations. The condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, implies that if G is $(3A, 3A, rZ)$-generated, then we must have $r > 3$. That is $r = 7$ or $r = 19$.

Proposition 3.6. *G is not $(3A, 3A, 7A)$-generated group.*

Proof. From Table 1 we can see that only $H_1 = H_2 \cong 7^2{:}GL(2,7){:}2$ are the only maximal subgroup of G that contains elements of orders 3 and 7 that have fusions into $3A$ and $7A$. We also know that $H_1 \bigcap H_2 \cong SL(2,7){:}2$, which has fusions into both $3A$ and $7A$. Now the computations give that $\Delta_G(3A,3A,7A) = 4802$, $\sum_{H_1}(3a,3a,7a) = \sum_{H_2}(3a,3a,7a) = 2744$, $\sum_{H_1 \bigcap H_2}(3a,3a,7a) = 14$, $h(7A, H_1 \bigcap H_2) = 49$. From Table 1, we also have $h(7A,H_1) = h(7A,H_2) = 8$. It follows that

$$\begin{aligned}\Delta^*_G(3A,3A,7A) &= \Delta_G(3A,3A,7A) - 8\times\sum_{H_1}(3a,3a,7a) - 8\times\sum_{H_2}(3a,3a,7a)\\ &+ 49\times\sum_{H_1\bigcap H_2}(3a,3a,7a) = 4802 - 43904 + 686\\ &< |C_G(7A)|\end{aligned}$$

where by $C_G(nX)$ we mean the centralizer of a representative of class nX of G. Showing the non-generation of G by $(3A,3A,7A)$. $\square$

Proposition 3.7. *G is $(3A,3A,7X), X \in S$-generated group.*

Proof. Here we have five maximal subgroups of G are involved, namely $H_1 = H_2$ and $H_3 \cong H_4 \cong H_5$. We know from the proof of Proposition 3.3 that $H_1 \bigcap H_2 \cong GL(2,7){:}2$, $H_1 \bigcap H_3 \cong D_{16}$, $H_1 \bigcap H_4 \cong D_{12}$, $H_1 \bigcap H_5 \cong 7{:}6$, and $H_1 \bigcap H_2 \bigcap H_5 \cong \mathbb{Z}_6$. The unique class of elements of order 7 in $H_1 \bigcap H_2 \cong SL(2,7){:}2$ fuses into the class $7A$ of G. We can see that the order of $H_1 \bigcap H_3 \cong D_{16}$, $H_1 \bigcap H_4 \cong D_{12}$ are both not divisible by 7, while the order of $H_1 \bigcap H_5 \cong 7{:}6$ is divisible by both 3 and 7. Also the order of $H_1 \bigcap H_2 \bigcap H_5 \cong \mathbb{Z}_6$ is neither divisible by 7. Note the intersections of H_1 also hold for H_2. We also noted that in H_3, $7a \to 7B$, in H_4, $7a \to 7C$ and H_5 we had $7a \to 7D$. We thus conclude that there will be no contribution from the intersections of H_1, H_2, H_3, H_4 and H_5 (pairwise or the three of them) in the computations of $\Delta^*_G(3A,3A,7X)$. Now the computations show that $\Delta_G(3A,3A,7X) = 1715$. Also $\sum_{H_1}(3a,3a,7x) = \sum_{H_2}(3a,3a,7x) = 686, x \in \{c,d,e\}$ and $\sum_{H_3}(3a,3a,7a) = \sum_{H_4}(3a,3a,7a) = \sum_{H_5}(3a,3a,7a) = 21$. We also have $\sum_{7:6}(3a,3a,7a) = 0$ From Table 1 we also have $h(7X,H_1) = h(7X,H_2) = 1$, $h(7X,H_3) = h(7X,H_4) = h(7X,H_5) = 7$, and $h(7D, H_1 \bigcap H_5) = 7$. It follows that for $X \in \{B,C\}$, $x \in \{c,d\}$ and $i \in \{3,4\}$ we get

$$\begin{aligned}\Delta^*_G(3A,3A,7B) &= \Delta_G(3A,3A,7B) - 1\times\sum_{H_1}(3a,3a,7c) - 1\times\sum_{H_2}(3a,3a,7c)\\ &- 7\times\sum_{H_3}(3a,3a,7a))\\ &= 1715 - 2\times 686 - 7\times 21 = 193 > 0,\end{aligned}$$

Similarly, keeping in mind the fusions, $\Delta^*_G(3A,3A,7C) = 193 > 0$. Also

$$\begin{aligned}\Delta^*_G(3A,3A,7D) &= \Delta_G(3A,3A,7D) - 1\times\sum_{H_1}(3a,3a,7e) - 1\times\sum_{H_2}(3a,3a,7e)\\ &- 7\times\sum_{H_5}(3a,3a,7a) + 7\times\sum_{7:6}(3a,3a,7a)\\ &= 1715 - 2\times 686 - 7\times 21 + 0 = 193 > 0,\end{aligned}$$

establishing the generation of G by $(3A,3A,7X),\ X \in S$. $\square$

Proposition 3.8. *G is $(3A,3A,19Y)$-generated group for all $Y \in T$.*

Proof. From Table 1 we can see that $H_8 = 19{:}3$ is the only maximal subgroup of G containing elements of order 19 and also has fusions into the class $3A$ of G. In addition to the identity class $1a$, the group H_8 has two classes of elements of order 3, namely $3a$ and $3b$; and has 6 conjugacy classes of elements of order 19, namely $19a$, $19b$, $19c$, $19d$, $19e$, and $19f$. Let $M := \{a,b,c,d,e,f\}$. For class $3a$ of H_8 and with the aid of GAP we found that $\sum_{H_8}(3a,3a,19x) = 0$ for all $x \in M$, while for class $3b$ of H_8 we found that $\sum_{H_8}(3b,3b,19x) = 0$ for all $x \in M$. We also found that $\Delta_G(3A,3A,19Y) = 1349$, for all $X \in T$. It follows that

$$\begin{aligned}\Delta^*_G(3A,3A,19X) &= \Delta_G(3A,3A,19Y)\\ &= 1349 > 0.\end{aligned}$$

Hence G is generated by all the triple $(3A, 3A, 19Y)$, for $Y \in T$. □

Next we turn to look at the $(3, 7, r)$-generations.

Proposition 3.9. *G is not $(3A, 7A, 7X)$-generated group for $X \in \{A, B\}$.*

Proof. The computations with GAP show that $\Delta_G(3A, 7A, 7A) = 343$ and from the Atlas [11] we can see that $|C_G(7A)| = 686$, where by $C_G(nX)$ we mean the centralizer of a representative of class nX of G. Now the non-generation of G by $(3A, 7A, 7A)$ follows by Lemma 2.7 of [6].

For the other case $(3A, 7A, 7X), X \in S$ we can see from Table 1 that only $H_1 = H_2 = 7^2{:}SL(2, 7){:}2$ are the maximal subgroups of G that have fusions into the classes $3A$, $7A$ and $7B$ of G. In fact each of H_1 and H_2 has $7a$ and $7b$ that fuse to class $7A$ of G. We also have that from H_1 and H_2 we have $7c \to 7B$, $7d \to 7C$ and $7e \to 7D$ and last one class of elements of order 3 that fuse to class $3A$ of G. The intersection of H_1 and H_2 has no element of order 7 that fuse to class $7X$ of G. Now the computations with GAP reveal $\Delta_G(3A, 7A, 7X) = 98$, Let $x \in \{c, d, e\}$ $\sum_{H_1}(3a, 7a, 7x) + \sum_{H_1}(3a, 7b, 7x) = 0 + 49 = 49$, $\sum_{H_2}(3a, 7a, 7x) + \sum_{H_2}(3a, 7b, 7x) = 0 + 49 = 49$. Also from Table 1 we have $h(7X, H_1) = h(7X, H_2) = 1$. Therefore we get

$$\begin{aligned}\Delta^*_G(3A, 7A, 7X) &= \Delta_G(3A, 7A, 7X) - 1 \times (\sum_{H_1}(3a, 7a, 7x) + \sum_{H_1}(3a, 7b, 7x)) \\ &- 1 \times (\sum_{H_2}(3a, 7a, 7xb) + \sum_{H_2}(3a, 7a, 7x)) = 98 - 49 - 49 = 0,\end{aligned}$$

showing the non-generation of G by $(3A, 7A, 7X)$ and completing the proof. □

Proposition 3.10. *G is $(3A, 7X, 7X)$-generated group, where $X \in S$.*

Proof. In this case five maximal subgroups are involved, namely, H_1, H_2 and also H_3, H_4 and H_5. For both H_1 and H_2 we have $7a, 7b$ fusing to $7A$. We have $7c \to 7B, 7d \to 7C$ and $7e \to 7D$. Also for H_3, H_4 and H_5 we have $7a \to 7B, 7a \to 7C$ and $7a \to 7D$ respectively. We also have, from $H_1 \bigcap H_5$, $7a \to 7D$. We also have We use the following computations to obtain $\Delta^*_G(3A, 7X, 7X)$. By GAP we have $\Delta_G(3A, 7X, 7X) = 882$, $\sum_{H_1}(3a, 7x, 7x) = 100$, $\sum_{H_2}(3a, 7x, 7x) = 98$, for $x \in \{c, d, e\}$ $\sum_{Hi}(3a, 7a, 7a) == 14,\ i = 3 \cdots 5$ Also $\sum_{H_1 \bigcap H_5}(3a, 7a, 7a) = 0$ We also have $H_7X, H_1) = h(7X, H_2) = 1$ and $h(7X, H_3) = 7$ and] $h(7D, H_1 \bigcap (H_5)) = 7$. It renders that

$$\begin{aligned}\Delta^*_G(3A, 7X, 7X) &= \Delta_G(3A, 7X, 7X) - 1 \times \sum_{H_1}(3a, 7x, 7x) - 1 \times \sum_{H_2}(3a, 7x, 7x) \\ &- 7 \times \sum_{Hi}(3a, 7a, 7a) = 882 - 98 - 98 - 7 \times 14 = 884 - 294 \\ &= 588 > 0.\end{aligned}$$

Hence G is $(3A, 7X, 7X)$-generated group. □

Corollary 3.11. *G is $(3A, 7W, 7X)$-generated group, where $W,\ X \in S, W \neq X$.*

Proof. This is the same as above only $\Delta_G(3A, 7W, 7X) = 1225$ and $\sum_{H_1}(3a, 7x, 7y) = \sum_{H_2}(3a, 7x, 7y) = 98, x, y \in \{c, d, e\}$.
For H_3, H_4 and H_5 for each there can only be fusion into one class and no two fuse to the same class. So there will be no contribution from these. So we have

$$\begin{aligned}\Delta^*_G(3A, 7W, 7X) &= \Delta_G(3A, 7W, 7X) - \sum_{H_1}(3a, 7x, 7y) - \sum_{H_2}(3a, 7x, 7y) \\ &= 1225 - 98m - 98 \\ &= 1029 > 0\end{aligned}$$

Hence G is $(3A, 7W, 7X)$-generated group. □

Proposition 3.12. *G is $(3A, 7X, 19Y)$-generated group for $X \in \{A, B, C, D\}$ and $Y \in T$.*

Proof. From Table 1 we can see that $H_8 = 19{:}3$ is the only maximal subgroup of G that contains elements of order 19. Clearly order of H_8 is not divisible by 7 and thus there is no contribution by any maximal subgroup of G in the computations of $\Delta^*_G(3A, 7X, 19Y)$ for $7A$ and $X \in S$ and $Y \in T$. That is $\Delta^*_G(3A, 7A, 19Y) = \Delta_G(3A, 7A, 19Y) = \Delta^*_G(3A, 7X, 19Y) = \Delta_G(3A, 7X, 19Y)$ for $7A \in 7X$ and $X \in S$ and $Y \in T$. Now the computations show that $\Delta_G(3A, 7A, 19Y) = 57$ and $\Delta_G(3A, 7X, 19Y) = 1064$ for $7A \in 7X$ and for all $X \in S$ and $Y \in T$. Hence G is $(3A, 7A, 19Y)$ and also $(3A, 7X, 19Y)$ -generated group for $7A \in 7X, X \in S$ and $Y \in T$. ☐

The last part of our investigation on the $(3, q, r)$-generations of G is to look at the $(3, 19, 19)$-generations, which is the context of the next proposition.

Proposition 3.13. *G is $(3A, 19Y, 19Z)$-generated group for all $Y,\ Z \in T$.*

Proof. The computations with GAP show that $\Delta_G(3A, 19Y, 19Z) = 2850$, for all $Y,\ Z \in T$. Again H_8 is the only maximal subgroup of G with classes that fuse to classes $3A$ and $19Y$ of G for $Y \in T$. By GAP we obtained that $\sum_{H_8}(3x, 19y, 19z) = 0$, for $x \in \{a, b\}$ and $y, z \in M$, where M as in the proof of Proposition 3.7. Thus $\Delta^*_G(3A, 19Y, 19Z) = \Delta_G(3A, 19Y, 19Z) = 2850$, for all $Y,\ Z \in T$. Hence G is generated by all the triples $(3A, 19Y, 19Z)$, for $Y,\ Z \in T$. ☐

3.3. The $(7, q, r)$- and $(19, q, r)$-generations of G

In this subsection we look at the $(7, q, r)$-generations of G, which comprise of the cases $(7, 7, r)$- and $(7, 19, r)$-generations.

Proposition 3.14. *G is neither $(7A, 7A, 7A)$ nor $(7A, 7A, 7X)$ - nor $(7A, 7A, 19Y)$-generated group for $X \in S$ and $Y \in T$.*

Proof. The GAP computations give $\Delta_G(7A, 7A, 7A) = 89$ and $\Delta_G(7A, 7A, 7X) = 14$. From the Atlas we can see that $|C_G(7A)| = 686$ and $|C_G(7X)| = 49$. Now the non-generation of G by $(7A, 7A, 7X)$ for $X \in S$ follows by Lemma 2.7 of [6].
For the other case of $(7A, 7A, 19Y)$, for $Y \in T$, the direct computations show that $\Delta_G(7A, 7A, 19Y) = 0$, for all $Y \in T$. Hence the result. ☐

Proposition 3.15. *G is $(7A, 7X, 7X)$-generated group and also $(7V, 7W, 7X)$, $V, W, X \in S$-generated group.*

Proof. The maximal subgroups of G with elements that fuse to class $7X$ of G are H_1, H_2 and H_3, H_4 and H_5, while those maximal subgroups with elements that fuse to both classes $7A$ and $7X$ of G are H_1 and H_2 only, where $7a, 7b \to 7A$. Now we consider the case $(7A, 7X, 7X)$ firstly. The intersection of H_1 and H_2 has no element of order 7 that fuse to any class $7X$ of G and thus $\sum_{H_1 \bigcap H_2}(7a, 7x, 7x) = 0$. Using GAP we get $\Delta_G(7A, 7X, 7X) = 118$, $\sum_{H_1}(7a, 7x, 7x) + \sum_{H_1}(7b, 7x, 7x) = 13 + 7 = 20$,. where $x \in \{c, d, e\}$ $\sum_{H_2}(7a, 7x, 7x) + \sum_{H_2}(7b, 7x, 7x) = 13 + 7 = 20$.
We also have $h(7X, H_1) = H_7X, H_2) = 1$. It follows that

$$\begin{aligned}
\Delta^*_G(7A, 7X, 7X) &= \Delta_G(7A, 7X, 7X) - 1 \times (\sum_{H_1}(7a, 7x, 7x) + \sum_{H_1} 7b, 7x, 7x)) \\
&- 1 \times (\sum_{H_2}(7a, 7x, 7x) + \sum_{H_2}(7b, 7x, 7x)) \\
&= 118 - 20 - 20 = 78 > 0,
\end{aligned}$$

showing the generation of G by $(7A, 7X, 7X)$.

We are now left with three cases namely $(7W, 7W, 7X)$, $(7W, 7W, 7X)$ and $(7V, 7W, 7X)$ the intersection of H_1 and H_2 has no element of order 7 that fuse to classes $7B$, $7C$ and $7D$ of G. The intersection of H_1 with either H_3 or H_4 has no element of order 7 at all. While $H_1 \bigcap H_5 \cong 7{:}6$ and has an element of order 7 that fuses to class $7D$ of G. The intersection of H_1, H_2 and H_5 is $\mathbb{Z}_6$. Also $\sum_{H_1 \bigcap H_5}(7a, 7a, 7a) = 5 < |C_{H_1| \bigcap H_5}(7a)$. Hence there is not generation that involves $H_1 \bigcap H_5$ by $(7a, 7a, 7a)$. Therefore there is no any contribution from the intersection of any subgroups of G in the computations of:

(i) $\Delta^*_G(7X, 7X, 7X)$

(ii) $\Delta^*_G(7X, 7X, 7Y)$

(iii) $\Delta^*_G(7X, 7Y, 7Z)$

$X, Y, Z \in S$. Using GAP we obtained:

(i) $\Delta_G(7X, 7X, 7X) = 1358$

(ii) $\Delta_G(7X, 7X, 7Y) = 602$

(iii) $\Delta_G(7X, 7Y, 7Z) = 1106 > 0$

Also noting that each of H_3, H_4 and H_5 has only one class $7a$ that fuses into $7B$, $7C$ and $7D$ of G respectively and also since $7a$ and $7b$ from H_1 and H_2 fuse into $7A$ in G and $7c$, $7d$ and $7e$ from each of H_1 and H_2 fuse into $7B$, $7C$ and $7D$ of G respectively we get:
$\sum_{H_1}(7x, 7x, 7x) = \sum_{H_2}(7x, 7x, 7x) = 35$, and $\sum_{H_i}(7x, 7x, 7x) < |C_{H_i}(7x)|, i = 1, 2,\ x \in \{c, d, e\}$. That is no contribution from $H_i(7x, 7x, 7x)$. Also $\sum_{H_3}(7a, 7a, 7a) = \sum_{H_4}(7a, 7a, 7a) = \sum_{H_5}(7a, 7a, 7a) = 12$. We also have $h(7B, H_3) = h(7C, H_4) = h(7D, H_5) = 7$. Therefore we get

$$\begin{aligned} \Delta^*_G(7X, 7X, 7X) &= \Delta_G(7X, 7X, 7X) - 7 \times \sum_{H_j}(7x, 7x, 7x) \\ &= 1358 - 7 \times 12 \\ &= 1358 - 84 = 1274 > 0. \end{aligned}$$

Hence G is $(7X, 7X, 7X)$-generated group.
Also G is $(7W, 7W, 7X)$ generated . First none of the $7a$ from H_3, H_4 and H_5 fuse into the same class. Hence there is no contribution from these on $\Delta_G(7W, 7W, 7X)$. Also we have $\Delta_G(7W, 7W, 7X) = 602$ and hence we have $\Delta^*_G(7X, 7X, 7X) = \Delta_G(7X, 7X, 7X) = 602$
Similarly $\Delta^*_G(7V, 7W, 7X) = \Delta_G(7V, 7W, 7X) = 1106$
The proposition is proved. □

Proposition 3.16. *G is $(7A, 7X, 19Y)$, $(7W, 7X, 19Y)$-generated group for $W, X \in S$ and $Y \in T$.*

Proof. For the triple $(7A, 7X, 19Y)$, we can see from Table 1 that $H_8 = 19{:}3$ is the only maximal subgroup of G containing elements of order 19. However H_8 does not contain elements of order 7. Thus there will be no contribution by any maximal subgroup of G in the computations of $\Delta^*_G(7A, 7X, 19Y)$ for $Y \in T$. That is $\Delta^*_G(7A, 7X, 19Y) = \Delta_G(7A, 7X, 19Y)$ for $Y \in T$. Now the computations show that $\Delta_G(7A, 7X, 19Y) = 57$ for all $Y \in T$. Hence G is $(7A, 7X, 19Y)$-generated group for $Y \in T$. For the cases $(7W, 7X, 19Y)$ we have :

The case $W = X$, we have $(7X, 7X, 19Z)$, $Z \in T$, we recall from Proposition 3.4 that G is $(2A, 7X, 19Z)$-generated group for all $Z \in T$. It follows by Lemma 2.5 of [6] that G is also $(7X, 7X, (19Z)^2)$-generated group for all $Z \in T$; that is G is $(7X, 7X, 19Y)$-generated group for all $Y \in T$, where $(19Z)^2 = 19Y$. Hence the result.

For the last case $W \neq X$, we have, $(7W, 7X, 19Y)$, $W, X \in S$, $Y \in T$, $W \neq X$, we have $\Delta_G(7W, 7X, 19Y) = 779$. Noting that there will be no contribution by any maximal subgroup of G. We have $\Delta^*_G(7W, 7X, 19Y) = \Delta_G(7A, 7X, 19Y) = 779$
Hence G is $(7W, 7X, 19Y)$ - generated. □

The last part of this subsection is to study the $(7, 19, r)$-generations of G.

Proposition 3.17. *For $Y \in T$, the group G is $(7A, 19Y, 19Y)$-generated, and also for $Y, Z \in T$ and $Y \neq Z$, the group G is $(7A, 19Y, 19Z)$-generated.*

Proof. The direct computations show that $\Delta_G(7A, 19Y, 19Y) = 114$, for all $Y \in T$. We know that H_8 is the only maximal subgroup of G that has elements of order 19. However it does not contains elements of orders 7. Thus there will be no contribution from any maximal subgroup of G in the computations of $\Delta^*_G(7A, 19Y, 19Z)$; that is $\Delta^*_G(7A, 19Y, 19Z) = \Delta_G(7A, 19Y, 19Z) = 114$, establishing the generation of G by $(7A, 19Y, 19Y)$ for $Y \in T$.
For the case $(7A, 19Y, 19Z)$, where $Y, Z \in T$ and $Y \neq Z$, the computations show that $\Delta_G(7A, 19Y, 19Z) = 171$. Like above, we know that H_8 is the only maximal subgroup of G that has elements of order 1. However it does not contains elements of orders 7. Thus there will be no contribution from any maximal subgroup of G in the computations of $\Delta^*_G(7A, 19Y, 19Z)$; that is $\Delta^*_G(7A, 19Y, 19Z) = \Delta_G(7A, 19Y, 19Z) = 171$, establishing the generation of G by $(7A, 19Y, 19Z)$ for $Y, Z \in T$ and $Y \neq Z$. □

Proposition 3.18. *G is $(7X, 19Y, 19Z)$ for all $X \in S$, and $Y, Z \in T$.*

Proof. As in the proof of Proposition 3.17, there will be no contribution from any maximal subgroup of G in the computations of $\Delta_G^*(7X, 19Y, 19Y)$; that is $\Delta_G^*(7X, 19Y, 19Y) = \Delta_G(7X, 19Y, 19Y)$. The computations with GAP show that $\Delta_G(7X, 19Y, 19Y) = 2128$ while $\Delta_G(5B, 19Y, 19Z) = 1995$, for $Y \neq Z$ and both Y and Z are in T. This establishes the generation of G by $(7X, 19Y, 19Z)$ for $Y, Z \in T$. □

Finally we handle the case $(19, q, r)$-generation of G. This comprises of only the case $(19, 19, 19)$.

Proposition 3.19. *G is $(19X, 19Y, 19Z)$-generated group for all $X, Y, Z \in T$.*

Proof. Using GAP we obtained that $\Delta_G(19X, 19X, 19X) = 4959,$. Now H_8 is the only maximal subgroup of G that has elements of order 19. It has 6 conjugacy classes of elements of order 19, where each class fuses into a class of elements of order 19 in G. Again the computations with GAP show that $\sum_{H_8}(19x, 19y, 19z) = 0$, $x, y, z \in \{a, b, c, d, e, f\}$. Since $h(19X, H_8) = 1$ for all $X \in T$, it follows that

$$\begin{aligned}\Delta_G^*(19X, 19Y, 19Z) &= \Delta_G(19X, 19Y, 19Z) - 1 \times \sum_{H_8}(19x, 19y, 19z)\\ &\in \{4949, 5645, 4959\}.\end{aligned}$$

Therefore $\Delta_G^*(19X, 19Y, 19Z) > 0$ and hence G is generated by all the triples $(19X, 19Y, 19Z)$, for all $X, Y, Z \in T$. □

4. The ranks of the classes of $PSL(3, 7)$

In this section we determine the ranks for all the non-trivial conjugacy classes of elements of the group $PSL(3, 7)$.

We start our investigation on the ranks of the non-trivial classes of $PSL(3, 7) := G$ by looking at the unique class of involutions $2A$. It is well-known that two involutions generate a dihedral group. Thus the lower bound of the rank of an involutry class in a finite group $G \neq D_{2n}$ (the dihedral group of order $2n$) is 3. The following proposition gives the rank of class $2A$ in G.

Proposition 4.1. *$rank(G{:}2A) = 3$.*

Proof. By Proposition 3.1 we have G is $(2A, 3A, 19Y)$-generated group, for all $Y \in T$, where T as in the previous section. It follows that by Lemma 2.3 of [6], G is $(2A, 2A, 2A, (19Y)^3)$-generated group; that is G is $(2A, 2A, 2A, 19Z)$-generated group, for some $Z \in T$. Therefore $rank(G : 2A) \leq 3$. Since $rank(G : 2A) \neq 2$, it follows that $rank(G : 2A) = 3$. □

Lemma 4.2. *$rank(G{:}7A) \neq 2$.*

Proof. For the class $7A$ of G, let

$$S = \{7A, 7B, 7C, 7D\}, \quad T = \{19A, 19B, 19C, 19D, 19E, 19F\} \text{ and } R = \{16A, 16B, 16C, 16D\}.$$

Direct computations show that $\Delta_G(7A, 7A, kX) = \Delta_G(7A, 7A, lY) = \Delta_G(7A, 7A, mZ) = 0$ for all $kX \in S$, $lY \in T$ and $mZ \in R$. Thus G is neither $(7A, 7A, kX)$- nor $(7A, 7A, lY)$- nor $(7A, 7A, mZ)$-generated group for all $kX \in S$, $lY \in T$ and $kZ \in R$. Also we have $(7A, 7, 2A) = 0$, so that G is also not a $(7A, 7A, 2A)$- generated group. Last, we laso have

$$\begin{aligned}\Delta_G(7A, 7A, 3A) &= 18 < 36 = |C_G(3A)|,\\ \Delta_G(7A, 7A, 4A) &= 8 < 16 = |C_G(2A)|,\\ \Delta_G(7A, 7A, 6A) &= 6 < 36 = |C_G(6A)|,\\ \Delta_G(7A, 7A, 8A) &= \Delta_G(7A, 7A, 8B) = 8 < 16 = |C_G(8A)| = |C_G(8B)|,\\ \Delta_G(7A, 7A, 14A) &= 7 < 14 = |C_G(14A)|,\end{aligned}$$

It follows that G cannot be generated by only two elements from class $7A$. □

Proposition 4.3. *$rank(G{:}7A) = 3$.*

Proof. From Proposition 4.2 we have shown that $rank(G{:}7A) \neq 2$. From Proposition 3.12 we have that G is $(3A, 7A, 19Y)$-generated group, $Y \in T$. It follows that by Lemma 2.3 of [6], G is $(7A, 7A, 7A, (19Y)^3)$-generated group; that is G is $(7A, 7A, 7A, 19Z)$-generated group where $Z \in T$. Hence $rank(G{:}7A) \leq 3$. That is $rank(G{:}7A) = 3$, since $rank(G{:}7A) \neq 2$. The proposition is proved. □

Proposition 4.4. *Let* $R = \{3A, 4A, 6A, 7B, 7C, 7D, 8A, 8B, 14A, 16A, 16B, 16C, 16D, 19A, 19B, 19C, 19D, 19E, 19F\}$. *Then* $rank(G{:}nX) = 2$ *for all* $nX \in R$.

Proof. The aim here is to show that G is $(nX, nX, 19A)$-generated group for all $nX \in R$. We recall from Table 1 that $H_8 = 19{:}3$ is the only maximal subgroup of G containing elements of order 19. Now for $nX \in R$, we give in Table 2 some information about $\Delta_G(nX, nX, 19A) := \Delta_G$, $h(19A, H_8)$, $\sum_{H_8}(nX, nX, 19A) := \sum_{H_8}$, and $\Delta^*_G(nX, nX, 19A) := \Delta^*_G$.

Table 2: Some information on the classes $nX \in S$

nX	Δ_G	$h(19A, H_8)$	$\sum_{H_8}$	$h(19A, H_8)\sum_{H_8}$	Δ^*_G
$3A$	2850	1	0	0	2850
$4A$	7182	1	0	0	7182
$6A$	12141	1	0	0	12141
$7B$	779	1	0	0	779
$7C$	779	1	0	0	779
$7D$	779	1	0	0	779
$8A$	7853	1	0	0	7853
$8B$	7853	1	0	0	7853
$14A$	9405	1	0	0	9405
$16A$	7353	1	0	0	7353
$16B$	7353	1	0	0	7353
$16C$	7353	1	0	0	7353
$16D$	7353	1	0	0	7353
$19A$	5645	1	2	2	5643
$19B$	5645	1	2	2	5643
$19C$	5645	1	2	2	5643
$19D$	5645	1	2	2	5643
$19E$	5645	1	2	2	5643
$19F$	5645	1	2	2	5643

The last column of Table 2 establishes the generation of G by the triple $(nX, nX, 19A)$ for all $nX \in R$. It follows that $rank(G{:}nX) = 2$ for all $nX \in R$. $\square$

Remark 4.5. For all $nX \in R$ of Proposition 4.4, it is possible show that G is $(2A, nX, 19A)$-generated group. Now the result follows by Corollary 2.6 of [6].

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References

[1] *The GAP Group gap – groups, algorithms, and programming, version 4.9.3.* `http://www.gap-system.org`. Accessed: 2018.

[2] A. R. Ashrafi, *Generating pairs for the Held group He*, J. Appl. Math. Comput., 10 (2002), pp. 167–174.

[3] ——, *(p, q, r)-generation of the sporadic group HN*, Taiwanese J. Math., 10 (2006), pp. 613–629.

[4] A. Basheer and T. T. Seretlo, *The (p, q, r)-generations of the mathieu group M_{22}*, Southeast Asian Bull. Math., 45 (2021), pp. 11–28.

[5] A. B. M. Basheer, *The ranks of the classes of A_{10}*, Bull. Iranian Math. Soc., 43 (2017), pp. 2125–2135.

[6] A. B. M. Basheer and J. Moori, *On the ranks of finite simple groups*, Khayyam J. Math., 2 (2016), pp. 18–24.

[7] ——, *On the ranks of the alternating group A_n*, Bull. Malays. Math. Sci. Soc., 42 (2019), pp. 1957–1973.

[8] A. B. M. Basheer and T. Seretlo, *On two generation methods for the simple linear group $PSL(3,5)$*, Khayyam J. Math., 5 (2019), pp. 125–139.

[9] ——, *The (p,q,r)-generations of the alternating group A_{10}*, Quaest. Math., 43 (2020), pp. 395–408.

[10] M. D. E. Conder, *Some results on quotients of triangle groups*, Bull. Austral. Math. Soc., 30 (1984), pp. 73–90.

[11] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, $\mathbb{ATLAS}$ *of finite groups*, Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[12] M. R. Darafsheh and A. R. Ashrafi, *Generating pairs for the sporadic group Ru*, J. Appl. Math. Comput., 12 (2003), pp. 143–154.

[13] M. R. Darafsheh, A. R. Ashrafi, and G. A. Moghani, *(p,q,r)-generations of the Conway group* Co_1 *for odd p*, Kumamoto J. Math., 14 (2001), pp. 1–20.

[14] ——, *(p,q,r)-generations and nX-complementary generations of the sporadic group Ly*, Kumamoto J. Math., 16 (2003), pp. 13–25.

[15] ——, *(p,q,r)-generations of the sporadic group O'N*, in Groups St. Andrews 2001 in Oxford. Vol. I, vol. 304 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2003, pp. 101–109.

[16] S. Ganeif and J. Moori, *(p,q,r)-generations and nX-complementary generations of the sporadic groups* HS *and* McL, J. Algebra, 188 (1997), pp. 531–546.

[17] S. Ganief and J. Moori, *2-generations of the smallest Fischer group* Fi_{22}, Nova J. Math. Game Theory Algebra, 6 (1997), pp. 127–145.

[18] ——, *(p,q,r)-generations of the smallest Conway group* Co_3, J. Algebra, 188 (1997), pp. 516–530.

[19] ——, *Generating pairs for the Conway groups* Co_2 *and* Co_3, J. Group Theory, 1 (1998), pp. 237–256.

[20] ——, *2-generations of the fourth Janko group J_4*, J. Algebra, 212 (1999), pp. 305–322.

[21] J. I. Hall and L. H. Soicher, *Presentations of some 3-transposition groups*, Comm. Algebra, 23 (1995), pp. 2517–2559.

[22] J. Moori, *Generating sets for F_{22} and its automorphism group*, J. Algebra, 159 (1993), pp. 488–499.

[23] ——, *(p,q,r)-generations for the Janko groups J_1 and J_2*, Nova J. Algebra Geom., 2 (1993), pp. 277–285.

[24] ——, *$(2,3,p)$-generations for the Fischer group F_{22}*, Comm. Algebra, 22 (1994), pp. 4597–4610.

[25] ——, *Subgroups of 3-transposition groups generated by four 3-transpositions*, Quaestiones Math., 17 (1994), pp. 83–94.

[26] ——, *On the ranks of the Fischer group F_{22}*, Math. Japon., 43 (1996), pp. 365–367.

[27] I. Zisser, *The covering numbers of the sporadic simple groups*, Israel J. Math., 67 (1989), pp. 217–224.

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