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Original Article

# Betterment for estimates of the numerical radii of Hilbert space operators

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**ABSTRACT:** We give several inequalities involving numerical radii $\omega(\cdot)$ and the usual operator norm $\|\cdot\|$ of Hilbert space operators. These inequalities lead to a considerable improvement in the well-known inequalities

$$\frac{1}{2}\|T\| \le \omega(T) \le \|T\|.$$

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## 1. Introduction and Preliminaries

Let $\mathbb{H}$ be a complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, and let $\mathbb{B}(\mathbb{H})$ be the $C^*$-algebra of all bounded linear operators acting on $\mathbb{H}$ equipped with the usual operator norm

$$\|T\| = \sup\{\|Tx\| : \ x \in \mathbb{H}, \|x\| = 1\}.$$

Throughout the paper, the symbol $\mathbf{1}_{\mathbb{H}}$ stands for the identity operator on $\mathbb{H}$. The absolute value of $T \in \mathbb{B}(\mathbb{H})$ is represented by $|T|$, and defined by $|T| = (T^*T)^{\frac{1}{2}}$, where $T^*$ is the adjoint operator of $T$. The usual operator norm fulfills the sub-multiplicativity property, i.e.,

$$\|S^*T\| \le \|S\|\,\|T\|; \ \ (S, T \in \mathbb{B}(\mathbb{H})). \tag{1}$$

An operator $T$ is said to be positive (denoted by $O \le T$) if $0 \le \langle Tx, x\rangle$ for all $x \in \mathbb{H}$. It is well-known that for $T \in \mathbb{B}(\mathbb{H})$, there is a unique partial isometry $U$ such as $T = U|T|$. Such a decomposition is called a polar decomposition of $T$.

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The numerical radius of $T \in \mathbb{B}(\mathbb{H})$, symbolized by $\omega(T)$, is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle| : \ x \in \mathbb{H}, \|x\| = 1\}.$$

It is easy to check that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm [9, Theorem 1.3-1]. More precisely, for $T \in \mathbb{B}(\mathbb{H})$ we have

$$\frac{1}{2}\|T\| \le \omega(T) \le \|T\|. \tag{2}$$

It is important to note that $\omega(\cdot)$ is not sub-multiplicative, but it is satisfies $\omega(S^*T) \le 4\omega(S)\omega(T)$ [9, Theorem 2.5-2].

We refer interested readers to [9] for the history and significance and [5, 7, 10, **?**] for recent developments in this field.

In 2003, Kittaneh [3] improved the second inequality in (2) by proving

$$\omega(T) \le \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{\frac{1}{2}}\right).$$

Two years later, in [4], the same author proved that

$$\frac{1}{4}\left\||T|^2 + |T^*|^2\right\| \le \omega^2(T) \le \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{3}$$

Notice that (3) improves both inequalities in (2).

Section 2 of this paper establishes a new improvement of the first and second inequality in (2). Our computations allow us to derive a modification of the triangle inequality for the usual operator norm.

## 2. Main Results

We start this section with the following simple lemma, the primary tool in our analysis.

**Lemma 2.1.** *Let $A \in \mathbb{B}(\mathbb{H})$ be a positive operator. Then for any $x \in \mathbb{H}$,*

$$\|Ax\|^2 + \|(\|A\|\mathbf{1}_{\mathbb{H}} - A)x\|^2 \le \|A\|^2\|x\|^2.$$

**Proof.** Let $x \in \mathbb{H}$. If $A \in \mathbb{B}(\mathbb{H})$ is a positive contraction (in the sense of $\|A\| \le 1$), we can write

$$\begin{aligned}
\|Ax\|^2 &= \langle Ax, Ax\rangle \\
&= \left\langle AA^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right\rangle \\
&\le \left\|AA^{\frac{1}{2}}x\right\|\left\|A^{\frac{1}{2}}x\right\| \\
&\le \|A\|\left\|A^{\frac{1}{2}}x\right\|^2 \\
&\le \left\|A^{\frac{1}{2}}x\right\|^2 \\
&= \langle Ax, x\rangle,
\end{aligned}$$

where the first inequality obeys from the Cauchy-Schwarz inequality, the second inequality obtained from the sub-multiplicativity property of the usual operator norm, and in the third inequality, we utilized the fact that $\|A\| \le 1$ for contraction $A$.

On the other hand, we have

$$\begin{aligned}
\|(\mathbf{1}_{\mathbb{H}} - A)x\|^2 &= \langle(\mathbf{1}_{\mathbb{H}} - A)x, (\mathbf{1}_{\mathbb{H}} - A)x\rangle \\
&= \|x\|^2 - 2\langle Ax, x\rangle + \|Ax\|^2 \\
&= \|x\|^2 - 2\langle Ax, x\rangle + 2\|Ax\|^2 - \|Ax\|^2.
\end{aligned}$$

Merging the above two relations implies

$$\|Ax\|^2 + \|(\mathbf{1}_{\mathbb{H}} - A)x\|^2 \le \|x\|^2. \tag{4}$$

If we substitute $A$ by $A/\|A\|$ , in (4), we get

$$\left\|\frac{A}{\|A\|}x\right\|^2 + \left\|\left(\mathbf{1}_{\mathbb{H}} - \frac{A}{\|A\|}\right)x\right\|^2 \leq \|x\|^2,$$

which can be written as

$$\|Ax\|^2 + \|(\|A\|\,\mathbf{1}_{\mathbb{H}} - A)\,x\|^2 \leq \|A\|^2\,\|x\|^2.$$

This completes the proof. □

The following result demonstrates Lemma 2.1 in a more general setting. More precisely, in the following theorem, the positivity of the operator is not needed.

**Theorem 2.2.** *Let $T \in \mathbb{B}(\mathbb{H})$ with the polar decomposition $T = U\,|T|$. Then for any $x \in \mathbb{H}$,*

$$\|T^*x\|^2 + \|(\|T\|\,U^* - T^*)\,x\|^2 \leq \|T\|^2\|x\|^2,$$

*and*

$$\|Tx\|^2 + \|(\|T\|\,U - T)\,x\|^2 \leq \|T\|^2\|x\|^2.$$

**Proof.** Let $T = U\,|T|$ be the polar decomposition of $T$ (of course, $T^* = |T|\,U^*$). Replacing $A$ and $x$ by $|T|$ and $U^*x$, respectively, in Lemma 2.1, we get

$$\begin{aligned}\|T^*x\|^2 + \|(\|T\|\,U^* - T^*)\,x\|^2 &= \||T|\,U^*x\|^2 + \|(\|\,|T|\,\| - |T|)\,U^*x\|^2\\ &\leq \|\,|T|\,\|^2\|U^*x\|^2\\ &= \|T\|^2\|U^*x\|^2\\ &\leq \|T\|^2\|x\|^2.\end{aligned}$$

This finishes the proof of the first inequality.

To prove the second inequality remember that if $T = U\,|T|$ is the polar decomposition of the operator $T \in \mathbb{B}(\mathbb{H})$, then $T^* = U^*\,|T^*|$ is also the polar decomposition of the operator $T^* \in \mathbb{B}(\mathbb{H})$ [1, p. 59]. If we replace $A$ and $x$ by $|T^*|$ and $Ux$, respectively, in Theorem 4, we infer that

$$\|Tx\|^2 + \|(\|T\|\,U - T)\,x\|^2 \leq \|T\|^2\|x\|^2$$

as desired. □

The following theorem nicely improves inequality (1).

**Theorem 2.3.** *Let $S, T \in \mathbb{B}(\mathbb{H})$ with the polar decompositions $S = U\,|S|$ and $T = V|T|$, respectively. Then for any $x \in \mathbb{H}$,*

$$\|S^*Tx\|^2 + \left\langle\left(\left|\,\|S\|\,U^*T - S^*T\right|^2 + \|S\|^2\left|\,\|T\|\,V - T\right|^2\right)x, x\right\rangle \leq \|S\|^2\|T\|^2\|x\|^2.$$

*In particular,*

$$\|S^*T\|^2 + \mu \leq \|S\|^2\|T\|^2,$$

*where*

$$\mu = \inf_{\substack{x\in\mathbb{H}\\ \|x\|=1}} \left\{\left\langle\left(\left|\,\|S\|\,U^*T - S^*T\right|^2 + \|S\|^2\left|\,\|T\|\,V - T\right|^2\right)x, x\right\rangle\right\}.$$

**Proof.** If we substitute $T$ by $S$ and $x$ by $Tx$ in Theorem 2.2, we conclude that

$$\begin{aligned}&\|S^*Tx\|^2 + \|(\|S\|\,U^*T - S^*T)\,x\|^2\\ &\leq \|S\|^2\|Tx\|^2\\ &\leq \|S\|^2\left(\|T\|^2\|x\|^2 - \|(\|T\|\,V - T)\,x\|^2\right) \quad \text{(by Theorem 2.2)}\end{aligned}$$

for any $x \in \mathbb{H}$. Thus,

$$\|S^*Tx\|^2 + \|(\|S\|\,U^*T - S^*T)\,x\|^2 + \|S\|^2\|(\|T\|\,V - T)\,x\|^2 \leq \|S\|^2\|T\|^2\|x\|^2.$$

This finishes the proof of the first inequality.

If $x \in \mathbb{H}$ is a unit vector, we can write from the first inequality

$$\|S^*Tx\|^2 + \mu \leq \|S\|^2 \|T\|^2.$$

Now, by taking supremum over all unit vector $x \in \mathbb{H}$, we reach

$$\|S^*T\|^2 + \mu \leq \|S\|^2 \|T\|^2$$

as expected. □

In the subsequent, we need the following two lemmas. The first lemma, which includes a mixed Schwarz inequality, can be seen in [2, pp. 75-76].

**Lemma 2.4.** *Let $T \in \mathbb{B}(\mathbb{H})$. Then*

$$|\langle Tx, y\rangle|^2 \leq \langle |T|\, x, x\rangle \langle |T^*|\, y, y\rangle\, ; \quad (x, y \in \mathbb{H})\,.$$

The second lemma known in the literature as the Hölder-McCarthy inequality tracks from the spectral theorem for positive operators and the famous Jensen's inequality [8, Theorem 1.4].

**Lemma 2.5.** *Let $T \in \mathbb{B}(\mathbb{H})$ be a positive operator. Then for any $r \geq 1$,*

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle\, ; \quad (x \in \mathbb{H}, \|x\| = 1)\,.$$

**Theorem 2.6.** *Let $T \in \mathbb{B}(\mathbb{H})$ with the polar decomposition $T = U\,|T|$. Then for any unit vectors $x, y \in \mathbb{H}$,*

$$|\langle Tx, y\rangle|^2 \leq \|T\|^2 - \sqrt{\lambda\gamma},$$

*where*

$$\lambda = \inf_{\substack{x\in\mathbb{H} \\ \|x\|=1}} \left\{ \left\langle \left| \, \|T\|\, U - T \right|^2 x, x \right\rangle \right\} \quad \text{and} \quad \gamma = \inf_{\substack{y\in\mathbb{H} \\ \|y\|=1}} \left\{ \left\langle \left| \, \|T\|\, U^* - T^* \right|^2 y, y \right\rangle \right\}.$$

**Proof.** It observes from Theorem 2.2 that

$$\left\langle |T|^2 x, x \right\rangle + \left\langle \left| \, \|T\|\, U - T \right|^2 x, x \right\rangle \leq \|T\|^2 \|x\|^2 \tag{5}$$

and

$$\left\langle |T^*|^2 y, y \right\rangle + \left\langle \left| \, \|T\|\, U^* - T^* \right|^2 y, y \right\rangle \leq \|T\|^2 \|y\|^2 \tag{6}$$

for any vectors $x, y \in \mathbb{H}$. Accordingly,

$$\left\langle |T|^2 x, x \right\rangle + \lambda \leq \|T\|^2 \quad \text{and} \quad \left\langle |T^*|^2 y, y \right\rangle + \gamma \leq \|T\|^2 \tag{7}$$

for any unit vectors $x, y \in \mathbb{H}$. Consequently,

$$\begin{aligned} |\langle Tx, y\rangle|^2 &\leq \langle |T|\, x, x\rangle \langle |T^*|\, y, y\rangle \\ &\leq \sqrt{\left\langle |T|^2 x, x \right\rangle \left\langle |T^*|^2 y, y \right\rangle} \\ &\leq \sqrt{\left( \|T\|^2 - \lambda \right) \left( \|T\|^2 - \gamma \right)} \\ &\leq \|T\|^2 - \sqrt{\lambda\gamma} \end{aligned}$$

where the first inequality and the second inequality follow from Lemma 2.4 and Lemma 2.5, respectively, and the last inequality is obtained from the arithmetic-geometric mean inequality (see [6, Lemma 4.1] for the details of its proof and its refinement). □

The following result modifies the second inequality in (2).

**Corollary 2.7.** *Let $T \in \mathbb{B}(\mathbb{H})$ with the polar decomposition $T = U|T|$. Then*

$$\omega^2(T) + \max\{\lambda, \gamma\} \le \|T\|^2,$$

*where*

$$\lambda = \inf_{\substack{x\in\mathbb{H}\\ \|x\|=1}} \left\{\left\langle \left| \|T\| U - T\right|^2 x, x\right\rangle\right\} \quad \text{and} \quad \gamma = \inf_{\substack{x\in\mathbb{H}\\ \|x\|=1}} \left\{\left\langle \left| \|T\| U^* - T^*\right|^2 x, x\right\rangle\right\}.$$

**Proof.** We have by Cauchy-Schwarz inequality,

$$|\langle Tx, x\rangle|^2 \le \|Tx\|^2 \quad \text{and} \quad |\langle T^*x, x\rangle|^2 \le \|T^*x\|^2$$

for any unit vector $x \in \mathbb{H}$. Hence, by (5) and (6), we reach

$$|\langle Tx, x\rangle|^2 + \left\langle |\|T\| U - T|^2 x, x\right\rangle \le \|T\|^2,$$

and

$$|\langle T^*x, x\rangle|^2 + \left\langle |\|T\| U^* - T^*|^2 x, x\right\rangle \le \|T\|^2.$$

From the overhead two inequalities, we have

$$|\langle Tx, x\rangle|^2 + \lambda \le \|T\|^2 \quad \text{and} \quad |\langle T^*x, x\rangle|^2 + \gamma \le \|T\|^2.$$

We obtain the expected result by taking supremum over all unit vectors $x \in \mathbb{H}$. □

**Remark 2.8.** *If, in Corollary 2.7, $T$ is a normal operator, then $\max\{\lambda, \gamma\} = 0$. This tracks from the point that $\omega(T) = \|T\|$ whenever $T$ is a normal operator [9, Theorem 1.4-2].*

The following result improves the triangle inequality for the usual operator norm.

**Corollary 2.9.** *Let $S, T \in \mathbb{B}(\mathbb{H})$ with the polar decompositions $S = U|S|$ and $T = V|T|$, respectively. Then*

$$\left\| |S^*|^2 + |T^*|^2 \right\| \le \|S\|^2 + \|T\|^2 - (\psi + \xi),$$

*where*

$$\psi = \inf_{\substack{x\in\mathbb{H}\\ \|x\|=1}} \left\{\left\langle |\|T\| V^* - T^*|^2 x, x\right\rangle\right\} \quad \text{and} \quad \xi = \inf_{\substack{x\in\mathbb{H}\\ \|x\|=1}} \left\{\left\langle |\|S\| U^* - S^*|^2 x, x\right\rangle\right\}.$$

*In particular,*

$$\left\| |T|^2 + |T^*|^2 \right\| \le 2\|T\|^2 - (\lambda + \gamma),$$

*where $\lambda$ and $\gamma$ are defined as in Theorem 2.6.*

**Proof.** It pursues from Theorem 2.2 that

$$\|S^*x\|^2 + \xi \le \|S\|^2 \quad \text{and} \quad \|T^*x\|^2 + \psi \le \|T\|^2$$

for any unit vector $x \in \mathbb{H}$. Therefore,

$$\begin{aligned} \left\langle \left(|S^*|^2 + |T^*|^2\right) x, x\right\rangle &= \left\langle |S^*|^2 x, x\right\rangle + \left\langle |T^*|^2 x, x\right\rangle \\ &= \|S^*x\|^2 + \|T^*x\|^2 \\ &\le \|S\|^2 + \|T\|^2 - (\psi + \xi). \end{aligned}$$

We get the desired result by taking supremum over all unit vectors $x \in \mathbb{H}$.
The second inequality can be received similarly through (7). □

The next result provides a refinement for the first inequality in (2), since

$$\frac{1}{2}\|T\| \le \frac{1}{2}\sqrt{\|T\|^2 + \max\{\lambda, \gamma\}} \le \omega(T).$$

**Corollary 2.10.** *Let $T \in \mathbb{B}(\mathbb{H})$ with the polar decomposition $T = U|T|$. Then*

$$\frac{1}{4}\|T\|^2 + \frac{1}{4}\max\{\lambda, \gamma\} \le \omega^2(T),$$

*where $\lambda$ and $\gamma$ are defined as in Theorem 2.6.*

**Proof.** It follows from (7) that

$$\|T^*x\|^2 + \gamma \le \|T\|^2 \le 4\omega^2(T)$$

for any unit vector $x \in \mathbb{H}$. Now by taking the supremum over $x \in \mathbb{H}$ with $\|x\| = 1$ in the above inequality we conclude that

$$\frac{1}{4}\left(\|T\|^2 + \gamma\right) \le \omega^2(T). \tag{8}$$

Likewise, we can show that

$$\frac{1}{4}\left(\|T\|^2 + \lambda\right) \le \omega^2(T). \tag{9}$$

Combining two inequalities (8) and (9) provides the expected inequality. □

**Remark 2.11.** *If $T^2 = O$, in Corollary 2.10, then $\max\{\lambda, \gamma\} = 0$. This follows from the fact that $\frac{1}{2}\|T\| = \omega(T)$ provided that $T^2 = O$ [3, Corollary 1].*

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