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Original Article

Classification of gyrogroups of orders at most 31

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ABSTRACT: A gyrogroup is defined as having a binary operation $\star$ containing an identity element such that each element has an inverse. Furthermore, for each pair (a, b) of elements of this structure, there exists an automorphism $\mathrm{gyr}[a, b]$ with the property that left associativity and the left loop property are satisfied. Since each gyrogroup is a left Bol loop, some results of Burn imply that all gyrogroups of orders $p, 2p$, and p^2, where p is a prime number, are groups. This paper aims to classify gyrogroups of orders 8, 12, 15, 18, 20, 21, and 28.

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(This work is dedicated to Professor Abraham Ungar for his pioneering role in gyrostructures)

1. Introduction

At the dawn of the 20th century, Albert Einstein changed the face of modern physics and astronomy by publishing four papers in 1905, also known as annus mirabilis papers [15]. In one of these papers, he introduced a new addition $\oplus$ on $\mathbb{R}_c^3$, the c-ball of $\mathbb{R}^3$ of all relativistically admissible, Einsteinian velocities, such that $(\mathbb{R}_c^3, \oplus)$ has an identity and each element of $\mathbb{R}_c^3$ has an inverse, but the associativity is not satisfied. Instead of associativity, it satisfies another condition called gyroassociative law. The theory of gyrostructures is a result of some innovative ideas developed by Abraham Ungar during the ninth decade of the 20th century [19, 20]. The gyrogroup is the closest algebraic structure to the group ever discovered.

In the first half of the twentieth century, by the impressions of geometry and algebra, Albert [1, 2], Baer [3] with more interest in geometry and Bruck [4] with more interest in algebra contributed to the theory of loops and quasigroups, which later appeared as an interpretation of Ungar's gyrogroups [19].

In 1989 Abraham Ungar continued his work to discover a pattern behind the seemingly lawless Einstein's addition of velocities [21]. He founded gyrogroup theory, non-associative group-like structures that share analogies

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with groups, and also provides a context for analytic hyperbolic geometry. In the past years, Ungar and other researchers proved many important results in gyrogroup theory and hyperbolic geometry, which have applications in the foundations of physics.

We are now ready to present the Ungar's definition of gyrogroup directly. According to the Ungar's famous books [22, 23], a pair $(G, \oplus)$ consists of a nonempty set G together with a binary operation $\oplus$ on G is called a *gyrogroup* if its binary operation satisfies the following axioms:

(G1) there exists an element $0 \in G$ such that for all $x \in G$, $0 \oplus x = x$;
(G2) for each $a \in G$, there exists $b \in G$ such that $b \oplus a = 0$;
(G3) there exists a function $\mathrm{gyr} : G \times G \longrightarrow \mathrm{Aut}\,(G)$ such that for every $a, b, c \in G$, $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$;
(G4) for each $a, b \in G$, $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$.

The automorphism $\mathrm{gyr}[a, b]$ is called the *gyroautomorphism generated by a and b*. A gyrogroup G can have another property, called gyrocommutativity, which is formulated as $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$, for all $a, b \in G$. A non-degenerate gyrogroup is a gyrogroup with a non-trivial gyroautomorphism. It is easy to see that a gyrogroup G is a group if and only if G does not have non-trivial gyroautomorphisms. This shows that a non-degenerate gyrogroup cannot be a group.

For the sake of completeness, we mention here a result which is crucial throughout this paper. The interested readers can consult [22, pp. 19-20] for its proof.

Theorem 1.1. *A magma $(G, \oplus)$ forms a gyrogroup if and only if it satisfies the following conditions:*

1. *there exists $0 \in G$ such that for all $a \in G$, we have $0 \oplus a = a = a \oplus 0$;*
2. *for each $a \in G$ there exists $b \in G$ such that $b \oplus a = 0 = a \oplus b$;*
3. *if we define $\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$, then $\mathrm{gyr}[a, b] \in \mathrm{Aut}\,(G, \oplus)$;*
4. *$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$;*
5. *$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c)$;*
6. *$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$;*
7. *$\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a]$.*

By Theorem 1.1, any gyroautomorphism is completely determined by its generators via the *gyrator identity*. Ungar discovered that the Möbius gyrogroup is a gyrocommutative gyrogroup and the set of all gyrations is not a subgroup of the whole automorphism group of the Möbius gyrogroup. In all known examples of finite gyrogroups before our investigations, the set of all gyrations of a finite gyrogroup forms a subgroup of the full automorphism group.

The main results of this paper are as follows:

Theorem 1.2. *There exists at least a gyrogroup of order 16 in which the set of all gyrations is not a group.*

Theorem 1.3. *Up to isomorphism, there are six non-degenerate gyrogroups of order 8; two non-degenerate gyrogroups of each order 12, 20, 28, and one non-degenerate gyrogroup of both orders 15, 21. There are no non-degenerate gyrogroups of orders a prime number, a prime square, two times of a prime number and 18. See Table 2 for more details.*

2. Classification of Non-Degenerate Gyrogroups of Orders ≤ 31

The aim of this section is to present the classification of gyrogroups of orders ≤ 31 except from the orders $24, 27$ and 30. Our classifications are based on Burn's results [5, 6, 8] and calculations given by Moorhouse [12].

2.1. An algorithm for constructing gyrogroups from Bol loops

Sabinin et al. [16] proved that all gyrogroups are left Bol loops and Burn [5] proved that all left Bol loops of orders p, $2p$ and p^2, p is prime, are groups. So, it is enough to consider all left Bol loops of a given order and then check the properties of a gyrogroup. By these results, we prepare an algorithm for constructing gyrogroups of a given order n based on the structure of all left Bol loops of order n. To see our algorithm, we choose a left Bol loop L of order n. For each $a, b \in L$, we define the mapping $\mathrm{gyr}[a, b] : L \longrightarrow L$ by $\mathrm{gyr}[a, b](x) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x))$ (the *gyrator identity*). If all mappings $\mathrm{gyr}[a, b]$ are automorphisms of L and these automorphisms satisfy the condition $G3$ then L will be a gyrogroup. Note that by [16, p. 13] the left loop property (G4) is equivalent to the left Bol identity and so we do not need to check the left loop property.

To construct all gyrogroups of a given order, it is enough to choose a left Bol loop K with binary operation $\oplus$. By Theorem 1.1(3), all gyrogroups satisfy the equation $\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$, where a, b and c are arbitrary elements of K. This implies that, if K is a gyrogroup then this equality should be satisfied. The following simple GAP code [9] define a function gyr[ ] based on three elements a, b and x from B.

Sub-Algorithm 1 Computing gyr[a,b](x)

```
Input: Elements a and b of a loop B;
Output: The function gyr[a, b] by definition given Theorem 1.1;
function GYR
    return ((a ∗ b)^{−1}) ∗ (a ∗ (b ∗ x));        ▷ g(a, b)(x) = −(a ⊕ b) ⊕ (a ⊕ (b ⊕ x))
end function
```

We now assume that A is the set of all gyrations gyr$[a, b]$, where a and b are elements of the left Bol loop K. Again, if K is a gyrogroup then Theorem [23, Theorem 1.13] shows that for each element $a \in K$, gyr$[a, a] =$ gyr$[a, -a] =$ gyr$[0, a] =$ gyr$[a, 0] = I$. Therefore, if one of these equalities are not satisfied, then K will not be a gyrogroup.

Sub-Algorithm 2 Computing the Non-Identity Gyroautomorphisms

```
Input: Two elements a and b of K.
Output: Add gyr[a, b] to A, when gyr[a, b] is a non-identity automorphism.
function GYROAUTO
    local m,a,b,x,y,z,A,B,C,D,DD,EK,EE,FF,KK
    A:=[ ]
    KK:= Difference(K,[Identity(K)])
    for a ∈ KK do
        for b ∈ KK do
            if a ≠ b and b ≠ a^{−1} then
                B:=[ ]
                C:=[ ]
                for x ∈ K do
                    Add(C,x)
                    Add(B,gyr(a,b,x))
                end for
                if B ≠ C then                      ▷ g(a,b) is non-identity permutation
                    Append(A,[[a,b],B])
                end if
            end if
        end for
    end for
    m:=Size(A)
    D:=Set(A[2,4..m])
    EE:=[ ]
    for y ∈ D do
        FF:=[ ]
        for z ∈ y do
            Add(FF,Position(K,z))
        end for
        Add(EE,FF)
    end for
    DD:=List(EE,x→PermList(x))
    if IsSubset(AutomorphismGroup(K),DD) then
        return A
    else
        return false
    end if
end function
```

By a result of Sabinin [16], all mappings gyr$[a, b]$ are bijective and so it is enough to check whether or not gyr$[a, b]$ is an automorphism.

Sub-Algorithm 3 Computing the Gyroautomorphisms Table of a Gyrogroup

```
function MATGYROAUTO
    local a,b,i,j,s,ss,t,n,m,x,y,z,M,B,D,DD,DDD,EE,FF
    M:=[ ]
    B:=["A","B","C","D","E","F","G","H","K","L","M","N","P",
       "Q","R","S","T","U","V","W","X","Y","Z","J","O"]
    m:=Size(A)
    D:=A[1,3..m-1]
    DD:=Set(A[2,4..m])
    EE:=[ ]
    for y ∈ DD do
        FF:=[ ]
        for z ∈ y do
            Add(FF,Position(K,z))
        end for
        Add(EE,FF)
    end for
    DDD:=List(EE,x⟶PermList(x))
    for a ∈ K do
        i:=Position(K,a)
        M[i]:=[ ]
        for b ∈ K do
            j:=Position(K,b)
            if  [a,b] ∈ D then
                t:=Position(A,[a,b])
                s:=A[t+1]
                ss:=Position(DD,s)
                M[i][j]:=B[ss]
            else
                M[i][j]:="I"
            end if
        end for
    end for
    Print("Non-identity automorphisms are as follows:",DDD)
    return M
end function
```

Sub-Algorithm 4 determines whether or not a left Bol loop K is a gyrogroup.

Sub-Algorithm 4 A function which shows that M is the Cayley table of a gyrogroup

```
function ISGYRO
    local k,a,b,c,n,A,B,N,K,KK
    if IsLoopTable(M)=false then
        return false
    end if                                              ▷ Check Loop Table
    K:=LoopByCayleyTable(M)
    KK:= Difference(K,[Identity(K)])
    for a ∈ KK do
        for b ∈ KK do
            if b ≠ a^{-1} then
                for c ∈ KK do
                    if a * (b * c) ≠ (a * b) * gyr[a, b]c then
                        return false
                    end if                              ▷ Check Gyroassociativity
                end for
            end if
```

```
        end for
    end for
    N:=gyroauto(K)                    ▷ Check the automorphism conditions for gyrations of M
    if N=false then
        return false
    else if N then=[ ]
        Print("This is a group")
    else
        Print("Cayley table and gyroautomorphism table")
        Print(matgyroauto(K,N))
    end if
end function
```

We end this subsection by introducing an example of a gyrogroup of order 16 in which the set of all gyroautomorphisms is not a group. To see this, we assume that G_{16} is a left Bol loop of order 16 with the Cayley table 1.

Table 1: The Cayley Table of G_{16}

$\oplus$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	4	6	2	7	3	5	10	15	8	12	11	14	13	9
2	2	3	5	7	1	6	0	4	12	11	14	10	15	8	9	13
3	3	2	7	5	0	4	1	6	11	12	13	15	10	9	8	14
4	4	6	1	0	5	3	7	2	14	13	12	8	9	10	15	11
5	5	7	6	4	3	0	2	1	15	10	9	14	13	12	11	8
6	6	4	0	1	7	2	5	3	13	14	11	9	8	15	10	12
7	7	5	3	2	6	1	4	0	9	8	15	13	14	11	12	10
8	8	9	11	12	13	15	14	10	5	7	1	6	4	3	2	0
9	9	8	12	11	14	10	13	15	7	5	0	4	6	2	3	1
10	10	15	13	14	11	9	12	8	1	0	5	3	2	6	4	7
11	11	12	8	9	10	14	15	13	2	3	4	0	1	7	5	6
12	12	11	9	8	15	13	10	14	3	2	6	1	0	5	7	4
13	13	14	15	10	9	12	8	11	6	4	3	7	5	0	1	2
14	14	13	10	15	8	11	9	12	4	6	2	5	7	1	0	3
15	15	10	14	13	12	8	11	9	0	1	7	2	3	4	6	5

Our GAP code shows that G_{16} is a gyrogroup and all non-identity gyroautomorphisms of G_{16} are as follows:

$(3,4)(5,7)(9,10)(11,16)$ $\qquad$ $(3,5)(4,7)(9,11)(10,16)$
$(3,5)(4,7)(9,16)(10,11)(12,13)(14,15)$ $\qquad$ $(3,7)(4,5)(9,11)(10,16)(12,13)(14,15)$
$(3,7)(4,5)(9,16)(10,11)$

It is an elementary fact that there is no elementary abelian group of order six and so we don't have a subgroup of $\mathrm{Aut}\,(G_{16})$. This proves Theorem 1.2. From the 1995 gyrogroups of order 16, there are 1180 gyrogroups such that their set of automorphisms does not form a group.

2.2. Gyrogroups of Orders 8, 12, 16, 18, 20, 21, 28

The aim of this subsection is to give a classification of gyrogroups of orders 8, 12, 16, 18, 20, 21, 28. Some gyrogroups of orders 27 are also given.

Mahdavi et al. [11] asked about the number of gyrogroups of order 8. The following lemma responds to this question.

Lemma 2.1. *There are exactly six gyrogroups of order 8.*

Proof. Burn [5, Theorem 6] proved that there are exactly six non-associative Bol loops of order 8. So, it is enough to check these Bol loops one by one. By our GAP code given in Subsection 2.1, it can be shown that all Bol loops of order 8 are gyrogroups and so there are exactly six gyrogroups of order 8. □

Table 2: The Number of All Non-Degenerate Gyrogroups and Non-Degenerate Gyrocommutative Gyrogroups

n	$\alpha(n)$	$\beta(n)$
8	6	3
12	2	0
15	1	1
16	1995	179
18	0	0
20	2	0
21	1	1
24	-	-
27	≥8	≥4
28	2	0
30	≥1	≥1

Lemma 2.2. *There are exactly two gyrogroups of each order 12, 20 and 28.*

Proof. By Burn [6, Theorems 1 and 2], there are exactly three non-associative Bol loops of order $4p$, p is prime, such that only one of them is a Moufang loop. Our calculations by the GAP code in Subsection 2.1 shows that the Moufang loop is not a gyrogroup, but both Bol loops which are not of Moufang type are gyrogroups. Therefore, there are exactly two gyrogroups of each order 12, 20 and 28. □

Lemma 2.3. *There is a unique gyrogroup of order 15 which is gyrocommutative.*

Proof. Niederreiter and Robinson [14] investigated the structure of Bol loops of order pq, where p and q are different primes and q divides $p^2 - 1$. As a consequence, there are exactly two Bol loops of order 15. Our GAP code shows that exactly one of these Bol loops of order 15 is a gyrogroup which is the gyrogroup reported by Suksumran in [17, p. 432]. □

Lemma 2.4. *There is no gyrogroup of order 18.*

Proof. Burn [8, 7, Theorem 6], proved that there are exactly two non-associative left Bol loop of order $2p^2$ which is not a Moufang loop. Our GAP code shows that for the case that $p = 3$, these Bol loops are not a gyrogroup. So, there is no gyrogroup of order 18. □

Lemma 2.5. *There is a unique gyrogroup of order 21 which is gyrocommutative.*

Proof. Kinyon et al. [10, Theorem 1.1], proved that if p and q are primes such that q divides $p^2 - 1$, then there exists a unique non-associative left Bruck loop of order pq, up to isomorphism, and there are precisely $\frac{(p-q+4)}{2}$ left Bol loops of order pq. This proves that there are exactly four Bol loops of order 21 that two of them are groups. Again we apply our GAP code to prove that precisely one of these non-associative Bol loops is a gyrogroup. □

Proof of Theorem 1.3. By Lemmas $2.1 - 2.5$, there are exactly six non-degenerate gyrogroups of order 8; two non-degenerate gyrogroups of each order 12, 20 and 28; one non-degenerate gyrogroup of both orders 15 and 21. There are no non-degenerate gyrogroups of orders a prime number, a prime square, two times of a prime number and 18.

There is no classification of Bol loops of orders 24, 27 and 30 and since our algorithm is based on the classification of Bol loops, we do not have all gyrogroups of these orders, but we can construct 8 gyrogroups of order 27 and one gyrogroup of order 30. There is no information about gyrogroups of order 24 in the literature. Finally, it is possible to construct 1995 gyrogroups of order 16. We record our results in Table 2. In this table, $\alpha(n)$ and $\beta(n)$ denote the number of all non-degenerate gyrogroups and non-degenerate gyrocommutative gyrogroups of order n, respectively.

3. Concluding Remarks

In this paper, a classification of gyrogroups of orders less than 32 except from 24, 27 and 30 are given. Our argument are based on Burn's classification of Bol loops [5, 6, 8] and calculations of Moorhouse [12]. Eight gyrogroups of order 27 and one gyrogroup of order 30 are also constructed. All tables of gyrogroups of orders 8, 12, 15, 16, 20, 21, 27, 28, and 30 are available from the corresponding author upon request.

Suppose G is a gyrogroup and $a, b \in G$. We define the commutator of a and b as $[a, b] = \ominus(a \oplus b) \oplus \mathrm{gyr}[a, b](b \oplus a)$. The subgyrogroup generated by all commutators of G, G', is called the derived subgyrogroup of G. Suksumran [18] proved that G' is a subgroup of G and conjectured that it is normal subgroup. We check this conjecture on all gyrogroups of orders less than 32 except from orders 24, 27 and 30 and we have the following result:

Proposition 3.1. *The commutator subgroup of all gyrogroups of orders less than 32 except from orders 24, 27 and 30 are normal in the whole gyrogroup.*

We end this paper with the following result that is important in finding gyrogroups of a given order n in the set of all Bol loops of order n.

Proposition 3.2. *All commutative gyrogroups of order < 81 are groups and there is a commutative gyrogroup of order 81 which is not a group.*

Proof. The commutative gyrogroups are precisely commutative Moufang loops and there is one commutative Moufang loop of order 81 which is not a group, see [13, p. 873] for details. Since the smallest non-associative commutative Moufang loop has order 81, all commutative gyrogroups of order < 81 are groups. □

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