AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 5(1) (2024) 39-44
https://doi.org/10.22060/AJMC.2023.21822.1112

Original Article

On l-reconstructibility of degree list of graphs

Rajab Ali Borzooei[*a], Mehrnoosh Shadravan[a]

[a] *Department of Mathematics, Shahid Beheshti University, Tehran, Iran*

ABSTRACT: The k-deck of a graph is the multiset of its subgraphs induced by k vertices which is denoted by $D_k(G)$. A graph or graph property is l-reconstructible if it is determined by the deck of subgraphs obtained by deleting l vertices. Manvel proved that from the $(n-l)$-deck of a graph and the numbers of vertices with degree i for all i, $n-l \leq i \leq n-1$, the degree list of the graph is determined. In this paper, we extend this result and prove that if G is a graph with n vertices, then from the $(n-l)$-deck of G and the numbers of vertices with degree i for all i, $n-l \leq i \leq n-3$, where $l \geq 4$ and $n \geq l+6$, the degree list of the graph is determined.

Review History:

Received:04 October 2022
Revised:18 February 2023
Accepted:11 March 2023
Available Online:01 January 2024

Keywords:

Reconstruction
l-Reconstructibility
Degree list

AMS Subject Classification (2010):

05C60; 05C07

1. Introduction

The well-known Graph Reconstruction Conjecture of Kelly [4, 5] and Ulam [14] has been open for more than 50 years. It asserts that every graph with at least three vertices can be (uniquely) reconstructed from its "deck" of vertex-deleted subgraphs. A card of a graph G is a subgraph of G obtained by deleting one vertex. The deck of G is the multiset of all cards of G. A graph is reconstructible if it is uniquely determined by its deck. Surveys on graph reconstruction include [2, 9].

Kelly [5] extended the conjecture, considering deletion of more than one vertex. A k-card of a graph is an induced subgraph having k vertices. The k-deck of G, denoted $D_k(G)$, is the multiset of all k-cards. Let G be a graph with n vertices. The graph G is k-deck reconstructible, if $D_k(G) = D_k(H)$ implies that $G \cong H$. The graph G is "l reconstructible" if it is determined by $D_{n-l}(G)$. The graph G is k-deck reconstructible and "l-reconstructible" have the same meaning when $k + l = n$. The reconstructibility of G, written $\rho(G)$, is the maximum l such that G is l-reconstructible.

The more general conjecture by Kelly [5] implies that for every positive integer l there exists M_l such that when $n \geq M_l$ every graph G with n vertices is determined by the $D_{n-l}(G)$. For a survey on this conjecture refer to [8].

There are several papers investigate what can be deduced about a graph from its k-deck. Manvel [10] proved for $n \geq 6$ that the $(n-2)$-deck of a graph with n vertices determines whether the graph satisfies the following

*Corresponding author.
E-mail addresses: borzooei@sbu.ac.ir, m_shadravan@sbu.ac.ir

properties: connected, acyclic, unicyclic, regular, and bipartite. Kostochka et al. [6] proved that connectedness is 3-reconstructible for graphs with n vertices when $n \geq 7$ (sharp by $\{C_5 + K_1, K''_{1,3}\}$ where $K''_{1,3}$ is the tree obtained from $K_{1,3}$ by subdividing two edges). Spinoza and West proved that connectedness of graphs with n vertices is l-reconstructible when $n \geq 2l^{(l+1)^2}$. Also, they showed that a complete r-partite graph is reconstructible from its $(r+1)$-deck. Kostochka et al. [7] proved that 3-regular graphs are 2 reconstructible. Some results about reconstruction have been extended to the context of reconstruction from the k-deck. For example, Bollobas [1] proved almost all graphs have reconstruction number 3. Spinoza and West [12] proved more generally that for $l = (1 - o(1))\frac{V(G)}{2}$ almost all graphs are l-reconstructible using only $\binom{l+2}{2}$ cards that omit l vertices. They also determined $\rho(G)$ exactly for every graph G with maximum degree at most 2. For more results on l-reconstructibility of graphs refer to [8, 11, 12].

Now, we concentrate on the results about l-reconstructibility of degree list of graphs. There are some of more important results in the following.

Theorem 1.1. *[3] For any graph with $n \geq 6$, the degree list is 2-reconstructible and this threshold on n is sharp.*

For sharpness, they considered $C_4 + k_1$ and $K'_{1,3}$.

Theorem 1.2. *[10] From the k-deck of a graph and the numbers of vertices with degree i for all i at least k, the degree list of the graph is determined.*

Theorem 1.3. *[10] The degree list of a graph G is reconstructible from $D_{\Delta(G)+2}(G)$.*

Taylor showed that the degree list is reconstructible from the k-deck when the number of vertices is not too much larger than k, regardless of the value of the maximum degree.

Theorem 1.4. *[13] If $l \geq 3$ and $n \geq g(l)$, then the degree list of any n-vertex graph is determined by its $(n-l)$-deck, where*

$$g(l) = (l + \log l + 1)(e + \frac{e \log l + e + 1}{(l-1)\log l - 1}) + 1$$

and e denotes the base of the natural logarithm. Thus the degree list is l-reconstructible when $e > el + O(\log(l))$.

Theorem 1.5. [7] *For $n \geq 7$, any two graphs of order n that have the same $(n-3)$-deck have the same degree list, and this threshold on n is sharp.*

For sharpness, they considered $C_5 + K_1$ and $k''_{1,3}$.

As remarked above, 2-reconstructibility and 3-reconstructibility of degree list of graphs are investigated in [3, 6]. So, we concentrate on $l \geq 4$ in this paper. Our goal is to extend the Theorem 1.2 for $l \geq 4$. The main theorem of this paper is stated as follows.

Theorem 1.6. *Let G be a graph with n vertices. Then, from the k-deck (l+k=n) of G and the numbers of vertices with degree i for all i, $k \leq i \leq n-3$ where $l \geq 4$ and $n \geq l+6$, the degree list of the graph G is determined.*

2. Main Results

Lemma 2.1. *[7] Let G be a graph with n vertices and a_j be the number of vertices of degree j in G. Denote by ϕ_i the total number of vertices of degree i over all cards in $D_k(G)$ (l=n-k) where $i \leq k-1$.*

$$\phi_i = \sum_{j=i}^{i+l} a_j \binom{j}{i}\binom{n-j-1}{l-j+i}. \tag{1}$$

Note that all of coefficients x, y, z and a, b, c and values n and l in the following lemmas are integer.

Lemma 2.2. *If $n \geq l+6$ and $l \geq 4$, then $\frac{1}{l}\binom{n-2}{l-1} > n$.*

Proof. It suffices to show that the following inequality holds:

$$(n-2)(n-3)\dots(n-l) > n \times (l)!.$$

We proceed by induction on n and l. The inequality is clearly true for $l = 4$ and $n \geq 10$ (the basis of the induction). Suppose that the inequality holds for l and n where $l \geq 4, n \geq 10$ and $n \geq l+6$. We show that it holds for $l+1$ and $n+1$.

By induction hypothesis, we have

$$(n-2)(n-3)\dots(n-l) > n \times (l)!.$$

So,

$$(n-1)(n-2)\dots(n-l) > n(n-1) \times (l)!.$$

Also, since $l \leq n-6$, we have

$$(n)(n-1) > (n+1) \times (l+1).$$

So, we have

$$(n-1)(n-2)\dots(n-l) > (n+1) \times (l+1)!.$$

□

Lemma 2.3. *If $n \geq l+6$ and $l \geq 3$, then $\frac{1}{l+1}\binom{n-2}{l} > n$.*

Proof. It suffices to show the following inequality holds:

$$(n-2)(n-3)\dots(n-l-1) > n \times (l+1)!.$$

We proceed by induction on n and l. The inequality is true for $l = 4$ and $n \geq 10$ (the basis of the induction). Suppose that the inequality holds for l and n where $l \geq 4, n \geq 10$ and $n \geq l+6$. We show that it holds for $l+1$ and $n+1$. By induction hypothesis, we have

$$(n-2)(n-3)\dots(n-l-1) > n \times (l+1)!.$$

So,

$$(n-1)(n-2)\dots(n-l-1) > n(n-1) \times (l+1)!.$$

Also, since $l \leq n-6$, we have

$$(n)(n-1) > (n+1) \times (l+2).$$

So, we have

$$(n-1)(n-2)\dots(n-l-1) > (n+1) \times (l+2)!.$$

□

Lemma 2.4. *If there exist $0 \leq x, y \leq n$ such that $x + y\binom{n-2}{l-1} = \binom{n-1}{l}$ where $n \geq l+6$ and $l \geq 4$. Then $x = 0$ and $y = \frac{1}{l}(n-1)$.*

Proof. By way of contradiction, assume $x > 0$. If $y = 0$, then $x = \binom{n-1}{l} > n$, a contradiction. So, suppose that $x, y > 0$. If $n-1 = al+b$ where $0 \leq b \leq l-1$, then there exist $a' > 0$ and $a'' \geq 0$ such that $y = a'$ and $x = (a'' + \frac{b}{l})\binom{n-2}{l-1}$ where $a' + a'' = a$. Since $x > 0$, we have $x \geq \frac{1}{l}\binom{n-2}{l-1}$. On the other hand, Lemma 2.2 implies that $\frac{1}{l}\binom{n-2}{l-1} > n$. So, $x > n$, a contradiction. □

Lemma 2.5. *Let $a + b\binom{n-2}{l-1}\binom{1}{1} = r$ such that $0 \leq a+b \leq n$ and $0 \leq a, b \leq n$, where $n \geq l+6$ and $l \geq 4$. If*

$$x + y\binom{n-2}{l-1}\binom{1}{1} = r,$$

where $0 \leq x, y \leq n$, then $x = a$ and $y = b$.

Proof. By way of contradiction, assume that $(x, y) \neq (a, b)$. Since $n \geq l+6$ and $l \geq 4$, we have $\binom{n-2}{l-1}\binom{1}{1} > n$. On the other hand, $(x-a)+(y-b)\binom{n-2}{l-1}\binom{1}{1} = 0$. Hence, $x = a+(b-y)\binom{n-2}{l-1}\binom{1}{1}$. If $(b-y) > 0$, then $x > \binom{n-2}{l-1}\binom{1}{1} > n$, a contradiction. If $(b-y) < 0$, then since $(b-y)\binom{n-2}{l-1}\binom{1}{1} < -n$, we have $x < a-n \leq 0$. Hence, $x < 0$, a contradiction. □

Lemma 2.6. *Let $a(l+1) + b\binom{n-2}{l} = r$ such that $0 \leq a+b \leq n$ and $0 \leq a, b \leq n$, where $n \geq l+6$ and $l \geq 4$. If*

$$x(l+1) + y\binom{n-2}{l} = r$$

where $0 \leq x, y \leq n$, then $x = a$ and $y = b$.

Proof. By contradiction, assume that $(x,y) \neq (a,b)$. Then Lemma 2.3 implies that $\binom{n-2}{l} > n(l+1)$. Also, $(x-a)(l+1) + (y-b)\binom{n-2}{l} = 0$. So, $x(l+1) = a(l+1) + (b-y)\binom{n-2}{l}$. If $b-y > 0$, then $x(l+1) > n(l+1)$. So, $x > n$, a contradiction. If $b-y < 0$, then $(b-y)\binom{n-2}{l} < -n(l+1)$. Also, $a(l+1) \leq n(l+1)$. So,

$$x(l+1) = a(l+1) + (b-y)\binom{n-2}{l} < 0.$$

Therefore, $x < 0$, a contradiction. □

Lemma 2.7. *Let $a + b\binom{n-2}{l-1}\binom{1}{1} + c\binom{n-1}{l}\binom{0}{0} = r$ such that $0 \leq a+b+c \leq n$ and $0 \leq a,b,c \leq n$ where $n \geq l+6$ and $l \geq 4$. If*

$$x + y\binom{n-2}{l-1}\binom{1}{1} + z\binom{n-1}{l}\binom{0}{0} = r,$$

where $0 \leq x,y,z \leq n$, then $x = a$.

Proof. If $z = c$, then by Lemma 2.5, we have $y = b$ and $x = a$. If $z \neq c$, then Lemma 2.4 implies that $x = a$. □

Theorem 2.8. *Let G be a graph with n vertices. Then from the k-deck $(l+k=n)$ of G and the numbers of vertices with degree i for all i, $k \leq i \leq n-3$ where $l \geq 4$ and $n \geq l+6$, the degree list of the graph is determined.*

Proof. Let r_1 be the total number of vertices of degree $k-1$ over all cards in $D_k(G)$. So, by (1), we have

$$\phi_{k-1} = a_{k-1}\binom{k-1}{0}\binom{l}{l} + a_k\binom{k}{1}\binom{l-1}{l-1} + \cdots + +a_{n-2}\binom{n-2}{l-1}\binom{1}{1} + a_{n-1}\binom{n-1}{l}\binom{0}{0} = r_1.$$

Also, we have a_i for all $k \leq i \leq n-3$ by hypothesis. Thus, we obtain a_{k-1} by Lemma 2.7. Let r_2 be the total number of vertices of degree $k-2$ over all cards in $D_k(G)$. By (1), we conclude that

$$\phi_{k-2} = a_{k-2}\binom{k-2}{0}\binom{l+1}{l} + a_{k-1}\binom{k-1}{1}\binom{l}{l-1} + \cdots + +a_{n-3}\binom{n-3}{l-1}\binom{2}{1} + a_{n-2}\binom{n-2}{l}\binom{1}{0} = r_2.$$

Moreover, we have a_i for all $k-1 \leq i \leq n-3$. Hence, we obtain a_{k-2} and a_{n-2} by Lemma 2.6. Also, by considering $\phi_{k-1} = r_1$, we obtain a_{n-1}. Now, we have a_i for all $k \leq i \leq n-1$. Therefore, by Theorem 1.2, the degree list is determined. □

Example 2.1. *Let G be a graph on 10 vertices with degree list (see Figure 1)*

$$(9,8,7,6,4,4,4,3,2,1).$$

Denote by a_i the number of vertices of degree i in G. We show that the degree list is determined from a_6, a_7 and $D_{n-4}(G)$. The number of vertices of degree 5 in $D_{n-4}(G)$ is 209. So, by (1), we have

$$\phi_5 = a_5\binom{5}{0}\binom{4}{4} + 1\binom{6}{1}\binom{3}{3} + 1\binom{7}{2}\binom{2}{2} + a_8\binom{8}{3}\binom{1}{1} + a_9\binom{9}{4}\binom{0}{0} = 209.$$

Now, one can easily prove that if there exist $0 \leq x,y,z \leq 10$ such that

$$x + 56y + 126z = 182,$$

then $x = 0$. So, $a_5 = 0$.

Also, the number of vertices of degree 4 in $D_{n-4}(G)$ is 200. Using (1), we imply that

$$\phi_6 = a_4\binom{4}{0}\binom{5}{4} + 0\binom{5}{1}\binom{4}{3} + 1\binom{6}{2}\binom{3}{2} + 1\binom{7}{3}\binom{2}{1} + a_8\binom{8}{4}\binom{1}{0} = 200.$$

Now, one can easily prove that if there exist $0 \leq x,y \leq 10$ such that

$$5x + 70y = 85,$$

then $x = 3$ and $y = 1$. So, $a_4 = 3$ and $a_8 = 1$.

Now, we obtain a_8 by $\phi_6 = 200$. Next, we obtain a_9 by $\phi_5 = 209$. Hence, by Lemma 1.2 the degree list is determined.

3. Conclusion

As we mentioned, it is proved that the degree list of graphs with at least 6 vertices is 2-reconstructible. Also, it is proved that the degree list of graphs with at least 7 vertices is 3-reconstructible. For the case $l = 4$, we showed that the degree list of a graph G is determined from the $(n-4)$-deck of G and the numbers of vertices with degree $n-4$ and $n-3$ when $n \geq 10$. By this result, 4-reconstructibility of the degree list of graphs can be investigated. As a future work, we will try to find n sufficiently large for which the degree list of graphs with n vertices is 4-reconstructible.

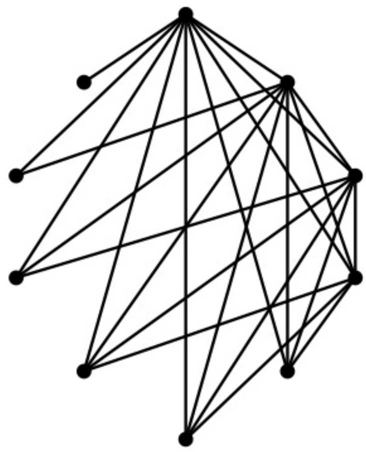

Figure 1: A graph with degree list $(9, 8, 7, 6, 4, 4, 4, 3, 2, 1)$.

References

[1] B. Bollobás, *Almost every graph has reconstruction number three*, J. Graph Theory, 14 (1990), pp. 1–4.

[2] J. A. Bondy and R. L. Hemminger, *Graph reconstruction—a survey*, J. Graph Theory, 1 (1977), pp. 227–268.

[3] Z. A. Chernyak, *Some additions to an article by B. Manvel: "Some basic observations on Kelly's conjecture for graphs"*, (Russian) Vestsī Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk, 126 (1982), pp. 44–49.

[4] P. J. Kelly, *ON ISOMETRIC TRANSFORMATIONS*, ProQuest LLC, Ann Arbor, MI, 1942. Thesis (Ph.D.)–The University of Wisconsin - Madison.

[5] P. J. Kelly, *A congruence theorem for trees*, Pacific J. Math., 7 (1957), pp. 961–968.

[6] A. V. Kostochka, M. Nahvi, D. B. West, and D. Zirlin, *Degree lists and connectedness are 3-reconstructible for graphs with at least seven vertices*, Graphs Combin., 36 (2020), pp. 491–501.

[7] ——, *3-regular graphs are 2-reconstructible*, European J. Combin., 91 (2021), Paper No. 103216, 10.

[8] A. V. Kostochka and D. B. West, *On reconstruction of graphs from the multiset of subgraphs obtained by deleting ℓ vertices*, IEEE Trans. Inf. Theory, 67 (2021), pp. 3278–3286.

[9] A. Maccari, O. Rueda, and V. Viazzi, *A survey on edge reconstruction of graphs*, J. Discrete Math. Sci. Cryptography, 5 (2002), pp. 1–11.

[10] B. Manvel, *Some basic observations on Kelly's conjecture for graphs*, Discrete Math., 8 (1974), pp. 181–185.

[11] D. Rivshin and S. a. P. Radziszowski, *Multi-vertex deletion graph reconstruction numbers*, J. Combin. Math. Combin. Comput., 78 (2011), pp. 303–321.

[12] H. Spinoza and D. B. West, *Reconstruction from the deck of k-vertex induced subgraphs*, J. Graph Theory, 90 (2019), pp. 497–522.

[13] R. Taylor, *Reconstructing degree sequences from k-vertex-deleted subgraphs*, Discrete Math., 79 (1990), pp. 207–213.

[14] S. M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York-London, 1960.

Please cite this article using:

Rajab Ali Borzooei, Mehrnoosh Shadravan, On l-reconstructibility of degree list of graphs, AUT J. Math. Comput., 5(1) (2024) 39-44
https://doi.org/10.22060/AJMC.2023.21822.1112