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Original Article

Generalized Ricci solitons on homogeneous Siklos space-times

Amirhesam Zaeim*[a], Jahangir Cheshmavar[a], Mohammad Ali Musavi[a]

[a]*Department of Mathematics, Payame Noor University, P.O. Box 19395-4697,Tehran, Iran*

ABSTRACT: In this paper, the class of homogeneous Siklos space-times is considered from algebraic point of view and the generalized Ricci solitons are completely classified.

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1. Introduction

Siklos space-times were introduced by Siklos in 1985 and he called these spaces “Lobatchevski plane waves” [29]. They were determined as exact solutions of the Einstein field equations and they are gravitational waves with a negative cosmological constant, propagating in the anti-de-Sitter universe [26]. Siklos space-times fall in the conformal class of pp-waves and appear in Petrov classification as space-times of type N [30]. One can assign a null non-twisting Killing field to all Siklos space-times. In vacuum, Siklos space-times belong to a special class of shear free solutions of Kundt type which are non-twisting and non-expanding [15].

With respect to the global coordinates $(x_1, x_2, x_3, x_4) = (v, u, x, y)$, Siklos metrics admit the following general form

$$g = -\frac{3}{\Lambda x_3^2}\left(2dx_1dx_2 + Hdx_2^2 + dx_3^2 + dx_4^2\right), \tag{1}$$

where $H = H(x_2, x_3, x_4)$ is an arbitrary smooth function ([25], [29]).

Siklos metrics are ubiquitous in mathematical physics and geometry. In particular,

**Corresponding author.*
E-mail addresses: zaeim@pnu.ac.ir, j_cheshmavar@pnu.ac.ir, musavi.mohammad.ali@student.pnu.ac.ir

- Siklos space-times can be considered as exact gravitational waves propagating in the anti-de Sitter universe [25].
- A classification of plane-fronted waves in space-times was proposed in [23], which depends on the sign of the cosmological constant Λ and a second-order invariant (determined by the sign of some constant k) that is related to the congruence of null rays. Siklos space-times appear in this classification as one of the two subcases with $\Lambda < 0$ and $k = 0$, and occur simultaneously with the subclass $(IV)_0$ of Kundt space-times.
- Siklos space-times are included in the study of all non-twisting type N solutions of Einstein field vacuum equations [2, 3]. A physical interpretation was obtained as a result of this general study by analysing the equation of geodesic deviation.
- Siklos metrics are studied as impulsive gravitational waves propagating in an anti-de-Sitter background in [27].
- The equations of vacuum polarization for photons which propagate in a general Siklos space-time were studied in [21], to explore the impact of one-loop vacuum polarization in the limit of geometric optics.

In addition, lots of special subclasses and representatives of Siklos space-times are significant and their physical interpretation and geometrical features have been examined in the literature; notable subclasses include Defrise spacetimes [14], Kaigodorov spacetimes [18] and generalized Defrise spacetimes [26]. Moreover, Ricci solitons ([5]-[7]), conformal geometry [10] and symmetries [9] were studied within the class of Siklos metrics.

The paper is organized as follows. In Section 2, we shall provide some basic concepts which will be applied in the future arguments. In Section 3, we take our attention to homogeneous metrics and we determine an algebraic description for the class of homogeneous Siklos space-times in the Section 4. Finally, Section 5 is devoted to a complete classification of generalized Ricci solitons on homogeneous Siklos space-times.

2. Preliminaries

In this section, we remind basics which are necessary for further arguments. These material consist of some information about homogeneous spaces and generalized Ricci solitons.

A (pseudo-)Riemannian homogeneous space (M, g) is a manifold M in which $I(M)$, the Lie group of isometries of (M, g), acts transitively on M. It is well known that a (pseudo-)Riemannian homogeneous space is diffeomorphic to a homogeneous space G/H where $G = I(M)$ and H is the isotropy subgroup at a point.

We denote the Lie algebra of G by $\mathfrak{g}$ and the isotropy subalgebra by $\mathfrak{h}$. We also denote the subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$ by the factor space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines its isotropy representation as

$$\phi : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m}), \qquad \phi(x)(y) = [x, y]_{\mathfrak{m}}, \quad \text{for all } \ x \in \mathfrak{g}, \ y \in \mathfrak{m}. \tag{2}$$

Let g be a matrix with respect to a basis $\{h_1, \cdots, h_r, u_1, \cdots, u_n\}$ of $\mathfrak{g}$, where $\{h_j\}$ and $\{u_i\}$ are bases for $\mathfrak{h}$ and $\mathfrak{m}$ where $1 \leq j \leq r = \dim H$ and $1 \leq i \leq n = \dim M$, respectively. The matrix g represents a bilinear form on $\mathfrak{m}$. Using the isotropy representation, a bilinear form is invariant if and only if ${}^t\phi(x) \cdot g + g \cdot \phi(x) = 0$, $\forall x \in \mathfrak{h}$, where $\cdot$ denotes matrix multiplication. It is well known (see [1], for instance) that invariant pseudo-Riemannian metrics $\bar{g}$ on the homogeneous space $M = G/H$ are in one-to-one correspondence with nondegenerate invariant symmetric bilinear forms g on $\mathfrak{m}$. Levi-Civita connection ∇ associated with the invariant bilinear form g is determined by the identity

$$\nabla(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + \nu(x, y), \qquad \text{for all } \ x, y \in \mathfrak{g}, \tag{3}$$

where $\nu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ is the $\mathfrak{h}$-invariant symmetric mapping uniquely determined by the following relation

$$2g(\nu(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}), \qquad \text{for all } x, y, z \in \mathfrak{g}.$$

The curvature tensor is then determined by

$$\begin{aligned} R : \mathfrak{m} \times \mathfrak{m} &\to \mathfrak{gl}(\mathfrak{m}) \\ (x, y) &\to [\nabla(x), \nabla(y)] - \nabla([x, y]). \end{aligned} \tag{4}$$

Finally, the Ricci tensor and the scalar curvature of g are deduced by the identities $\varrho = \operatorname{tr}(z \mapsto R(z, x)y)$ and $\tau = \operatorname{tr}_g \varrho$, respectively.

A natural generalization of Einstein metrics are Ricci solitons. By a Ricci soliton, we mean a (pseudo-)Riemannian manifold (M, g) with a smooth vector field V such that

$$\mathcal{L}_V g + \varrho = \lambda g,$$

where $\mathcal{L}$ denotes to the Lie derivative. Shrinking, steady and expanding Ricci solitons are the ones with $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Ricci solitons yield to the self-similar solutions of the Ricci flow equation and are critic in understanding its singularities. For a survey on Ricci solitons and their geometric aspects we may refer to [12]. For their wide applications in physics, Ricci solitons have been the subject of many studied, e.g., Ricci solitons on Kenmotsu manifolds in [16], Ricci solitons on a generalized Sasakian space form in [24], complete classification of Ricci and Yamabe solitons on non-reductive homogeneous spaces of dimension four in [11], Ricci solitons on Einstein-like neutral Lie groups of dimension four in [17], Ricci soliton on a nilpotent Lie group equipped with a left invariant Riemannian metric and on solvmanifolds in [19, 20]. Ricci solitons on conformally flat metrics of Siklos type in [6] and algebraic Ricci solitons on four-dimensional non-reductive homogeneous spaces in [8].

A generalization of Ricci solitons was introduced by Nurowski and Randall in [22]. In fact, the (pseudo-)Riemannian manifold (M, g) is called a generalized Ricci soliton whenever a smooth vector field V exists such that

$$\mathcal{L}_V g + 2\eta V^\flat \odot V^\flat - 2\beta\varrho = 2\lambda g, \tag{5}$$

for some real constants η, β and λ, where $V^\flat$ denotes the one-form metrically equivalent to V. Several important equations are deduced for special values of the coefficients in (5), e.g.,

- (K) the equation for Killing vector fields when $\eta = \beta = \lambda = 0$;
- (H) the equation for homothetic vector fields when $\eta = \beta = 0$;
- (RS) the quation of Ricci solitons when $\eta = 0$ and $\beta = 1$;
- (E-W) in conformal geometry, a special case of the Einstein-Weyl equation when $\eta = 1$ and $\beta = -\frac{1}{n-2}$, $(n > 2)$ [4];
- (PS) the equation for a metric projective structure with a skew-symmetric Ricci tensor representative in the projective class when $\eta = 1$, $\beta = -\frac{1}{n-1}$ and $\lambda = 0$ [28];
- (VN-H) the vacuum near-horizon geometry equation of a space-time when $\eta = 1$ and $\beta = \frac{1}{2}$, with λ playing the role of the cosmological constant [13].

3. Homogeneous Siklos space-times

In order to study homogeneous classes of a (pseudo-)Riemannian manifold, it is inevitable to study the generators of the isometry group, i.e, Killing vector fields of the space. With respect to the global coordinates $(x_1, \cdots, x_4)$, Killing vector fields of the Siklos space-times are studied and classified in eight classes $K_1, \cdots, K_8$, [29].

$$\begin{aligned}
&K_1 = \partial_1, \quad K_2 = -x_1\partial_1 + x_2\partial_2, \qquad K_3 = \partial_2, \qquad K_4 = \partial_4,\\
&K_5 = x_4\partial_1 - x_2\partial_4, \qquad K_{6,\alpha} = (2+\alpha)x_1\partial_1 + (2-\alpha)x_2\partial_2 + 2x_3\partial_3 + 2x_4\partial_4,\\
&K_7 = x_1\partial_1 - x_2\partial_2 + 2\partial_4, \qquad K_8 = -\tfrac{1}{2}(x_3^2 + x_4^2)\partial_1 + x_2^2\partial_2 + x_2x_3\partial_3 + x_2x_4\partial_4,
\end{aligned}$$

where α is an integer.

Clearly, more Killing vector fields give rise more homogeneity on the space. Different combinations of Killing vector fields were studied in [29] and we resume this classification in Table 1. Here, β is a real parameter and $A_\alpha(x_i)$ represents a homogeneous function of degree α.

We study homogeneous Siklos space-times as the spaces admitting at least five families of Killing vector fields. To this end, we study different cases as mentioned in Table 1 and evaluate possible homogenous types.

We start with the case 1), $H = x_2^{-2}A(x_3, x_4)$, and arrange more possible Killing vector fields. In this case K_1, K_2 are Killing vector fields. Clearly, adding K_3 will conclude $A(x_3, x_4) = 0$ which is not acceptable.

If K_4 is added, then $A(x_3, x_4) = A(x_3)$ and so K_5 and K_7 will be Killing vector fields automatically. We note here that $K_7 = 2K_4 - K_2$, and since K_2, K_4 are assumed to be Killing vector fields we discard K_7. Then we have

- If $K_{6,\alpha}$ is joined, then $A(x_3) = c_1x_3^2$, where c_1 is an arbitrary constant. Adding K_8 will give $A(x_3) = 0$ which is not desired.

Defining function	Basis of Killing vector fields
0) $H(x_2, x_3, x_4)$	K_1
1) $x_2^{-2}A(x_3, x_4)$	K_1, K_2
2) $A(x_3, x_4)$	K_1, K_3
3) $A_2(x_2, x_3, x_4)$	$K_1, K_{6,2}$
4) $A(x_2, x_3)$	K_1, K_4, K_5
5) $A_\alpha(x_3, x_4), \alpha \neq -2$	$K_1, K_3, K_{6,\alpha}$
6) $A(x_3)e^{x_4}$	K_1, K_3, K_7
7) $A(x_3)$	K_1, K_3, K_4, K_5
8) $A(x_2)x_3^2$	$K_1, K_4, K_5, K_{6,2}$
9) $x_2^{-2\beta-2}A(x_2^\beta x_3)$	$K_1, K_4, K_5, K_{6,\alpha}, \quad \alpha = 2(1 + \frac{1}{\beta})$
10) $A_{-2}(x_3, x_4)$	$K_1, K_3, K_{6,-2}, K_8$
11) $\pm x_3^\alpha, \alpha \neq -2$	$K_1, K_3, K_4, K_5, K_{6,\alpha}$
12) $\pm x_3^{-2}$	$K_1, K_3, K_4, K_5, K_{6,-2}, K_8$

Table 1: Killing vector fields of Siklos space-times

- If K_8 is joined, then $A(x_3) = c_1$, where c_1 is an arbitrary constant.

If K_5 or K_7 is joined to the case 1), then $A(x_3, x_4) = A(x_3)$ and the above arguments apply again.

If we add $K_{6,\alpha}$ to the case 1), then $A(x_3, x_4) = x_3^2 A_\alpha(x_3, x_4)$, more Killing vector fields (i.e., K_7 or K_8) will take us back to the previous arguments.

Let us study the case 2), $H = A(x_3, x_4)$. Adding K_2 gives $A = 0$. If K_4 is added to this case then K_5 will be a Killing vector field and we are in fact in the case 7) which will be studied later. If $K_{6,\alpha}$ or K_7 is added, then we are in the cases 5) and 6), respectively. If K_8 is added then we are in the case 12).

In the case 3), $H = A_2(x_2, x_3, x_4)$, then K_1 and $K_{6,2}$ are Killing vector fields. If K_7 is added then K_2, K_4 and K_5 will be Killing vector fields automatically, $H = c_1 x_2^{-2} x_3^2$ and more Killing vector fields is not possible. If K_8 be added to the case 3), then $H = x_2^{-4} A_2(x_3, x_4)$ and more Killing vector fields will be considered in the case 8).

In the case 4), $H = A(x_2, x_3)$ and K_1, K_4, K_5 are Killing vector fields. If we add K_2 or K_3, then possible outcomes were considered in the cases 1) and 2), respectively. Adding $K_{6,\alpha}$ will coincide with the case 9). If K_7 is added, then $A = x_2^{-2} A(x_3)$ and K_2 is a Killing vector field automatically. This case was considered before. Adding K_8 will bring us to the case 12).

In the case 5), $H = A_\alpha(x_3, x_4), \alpha \neq -2$ and $K_1, K_3, K_{6,\alpha}$ are Killing vector fields. If K_4 is added, then K_5 will be a Killing vector field automatically and we are in the case 11). If K_7 or K_8 is added, then $A = 0$ which is not desired.

In the case 6), $H = A(x_3)e^{x_4}$ and K_1, K_3, K_7 are Killing vector fields. Adding extra Killing vector fields is not possible.

In the case 7), $H = A(x_3)$ and K_1, K_3, K_4, K_5 are Killing vector fields. Adding $K_{6,\alpha}$ will take us to the cases 11) and 12) to be considered later. If K_7 is added, then $A(x_3) = 0$ and adding K_8 will be considered in the case 12).

In the case 8), $H = A(x_2)x_3^2$ and $K_1, K_4, K_5, K_{6,2}$ are Killing vector fields. Adding K_2 was considered in the case 1). If we add K_3 to this case, then we will be in a position to be explored in the case 11). Adding K_7 was discussed in the case 3). If K_8 is added to this case, then $A = c_1 x_2^{-4}$.

In the case 9), $H = x_2^{-2\beta-2} A(x_2^\beta x_3)$ and $K_1, K_4, K_5, K_{6,\alpha}$ for $\alpha = 2(1 + \frac{1}{\beta})$ are Killing vector fields. If K_2 is added to this case, then K_7 is a Killing vector field automatically and $A = c_1(x_2^\beta x_3)^2$. More Killing vector fields could not be added. Adding K_3 will bring us back to the case 7). If K_8 is added to this case, then $A = c_1(x_2^\beta x_3)^{\frac{2\beta}{\beta+1}}$

and K_7 is a Killing vector field automatically.

In the case 10), $H = A_{-2}(x_3, x_4)$ and $K_1, K_3, K_{6,-2}, K_8$ are Killing vector fields. Adding K_2 is not possible. If we add K_4, then K_5 is also a Killing vector field and $H = c_1 x_3^{-2}$. More Killing vector fields cannot be added.

In the case 11), $H = \pm x_3^{\alpha}, \alpha \neq -2$ and $K_1, K_3, K_4, K_5, K_{6,\alpha}$ are Killing vector fields. If we add K_8, then α has to be 2 and we are in a position to be explored in the case 12). Adding extra Killing vector fields is not possible.

We summarize the above arguments in Table 2.

Defining function	Basis of Killing vector fields
1) $H = \pm x_2^{-2} x_3^2$	$K_1, K_2, K_4, K_5, K_{6,\alpha}$
2) $H = \pm x_2^{-2}$	K_1, K_2, K_4, K_5, K_8
3) $H = \pm x_2^{-4} x_3^2$	$K_1, K_4, K_5, K_{6,2}, K_8$
4) $H = \pm x_2^{\frac{-4\beta-2}{\beta+1}} x_3^{\frac{2\beta}{\beta+1}}$	$K_1, K_4, K_5, K_{6,\alpha}, K_8, \alpha = 2(1+\frac{1}{\beta})$
5) $H = \pm x_3^{\alpha}, \alpha \neq -2$	$K_1, K_3, K_4, K_5, K_{6,\alpha}$
6) $H = \pm x_3^{-2}$	$K_1, K_3, K_4, K_5, K_{6,-2}, K_8$

Table 2: Siklos space-times with extra Killing vector fields.

Now, we study case by case, classes with extra Killing vector fields of the Table 2, in order to identify the algebraic description of homogeneous classes of the Siklos space-times.

Let $H = \pm x_2^{-2} x_3^2$, if set $e_1 = K_1, e_2 = K_2, e_3 = K_4, e_4 = K_5, e_5 = K_{6,\alpha}$, then the Lie algebra $\mathfrak{g}$ of Killing vector fields is generated by the following non-zero commutators

$$\begin{array}{lll} [e_1, e_2] = -e_1, & [e_1, e_5] = (2+\alpha)e_1, & [e_2, e_4] = e_4, \\ [e_3, e_4] = e_1, & [e_3, e_5] = 2e_3, & [e_4, e_5] = \alpha e_4. \end{array}$$

Now, by direct calculations, the isotropy subalgebra around the origin point $(0, 0, 1, 0)$ is generated by $\{e_2, e_4\}$ which shows that the underlying coset representation is three dimensional and so this case is not relevant to our study.

By similar arguments, one can determine the Lie algebra $\mathfrak{g}$ of Killing vector fields and the isotropy subalgebra $\mathfrak{h}$ around the point $(0, 0, 1, 0)$ as following.

Let $H = \pm x_2^{-2}$, if set $e_1 = K_1, e_2 = K_2, e_3 = K_4, e_4 = K_5, e_5 = K_8$, then $\mathfrak{g}$ is generated by the following non-zero commutators

$$\begin{array}{lll} [e_1, e_2] = -e_1, & [e_2, e_4] = e_4, & [e_2, e_5] = e_5, \\ [e_3, e_4] = e_1, & [e_3, e_5] = -e_4 & \end{array}$$

and $\mathfrak{h}$ is spanned by $\{e_1 + 2e_5, e_2, e_4\}$, so the coset representation is of dimension 2 which is not relevant to our study.

Let $H = \pm x_2^{-4} x_3^2$, if set $e_1 = K_1, e_2 = K_4, e_3 = K_5, e_4 = K_{6,2}, e_5 = K_8$, then the Lie algebra $\mathfrak{g}$ is generated by

$$\begin{array}{lll} [e_1, e_4] = 4e_1, & [e_2, e_3] = e_1, & [e_2, e_4] = 2e_2, \\ [e_2, e_5] = -e_3, & [e_3, e_4] = 2e_3, & \end{array}$$

and the isotropy subalgebra $\mathfrak{h} = \mathrm{span}\{e_1 + 2e_5, e_3\}$, thus the factor space is three dimensional which is not relevant to this study.

Let $H = \pm x_2^{\frac{-4\beta-2}{\beta+1}} x_3^{\frac{2\beta}{\beta+1}}$ and if set $e_1 = K_1, e_2 = K_4, e_3 = K_5, e_4 = K_{6,\alpha}, e_5 = K_8, \alpha = 2(1+\frac{1}{\beta})$, then the Lie algebra $\mathfrak{g}$ is generated by

$$\begin{array}{lll} [e_1, e_4] = \frac{2(1+2\beta)}{\beta} e_1, & [e_2, e_3] = e_1, & [e_2, e_4] = 2e_2, \\ [e_2, e_5] = -e_3, & [e_3, e_4] = \frac{2(\beta+1)}{\beta} e_3, & [e_4, e_5] = -\frac{2}{\beta} e_5. \end{array}$$

In this case, the isotropy subalgebra $\mathfrak{h}$ is spanned by $\{e_1 + 2e_5, e_3\}$, so the factor space is of dimension three which is not relevant to our study.

Let $H = \pm x_3^{\alpha}, \quad \alpha \neq -2$, then we set $\{e_1 = K_1, e_2 = K_3, e_3 = K_4, e_4 = K_5, e_5 = K_{6,\alpha}$ and the Lie algebra $\mathfrak{g}$ is generated by

$$\begin{array}{llll} [e_1, e_5] = (2+\alpha)e_1, & [e_2, e_4] = -e_3, & [e_2, e_5] = (2-\alpha)e_2, & \\ [e_3, e_4] = e_1, & [e_3, e_5] = 2e_3, & [e_4, e_5] = \alpha e4, & \alpha \neq -2. \end{array} \tag{6}$$

Then, the isotropy subalgebra $\mathfrak{h}$ is generated by e_4, so the factor space is of dimension four.

Finally, let $H = \pm x_3^{-2}$ and if set $\{e_1 = K_1, e_2 = K_3, e_3 = K_4, e_4 = K_5, e_5 = K_{6,-2}, e_6 = K_8\}$, then the Lie algebra $\mathfrak{g}$ is generated by the following non-zero commutators

$$\begin{array}{lll} [e_2, e_4] = -e_3, & [e_2, e_5] = 4e_2, & [e_2, e_6] = \frac{1}{2}e_5, \\ [e_3, e_4] = e_1, & [e_3, e_5] = 2e_3, & [e_3, e_6] = -e_4, \\ [e_4, e_5] = -2e_4, & [e_5, e_6] = 4e_6. & \end{array} \tag{7}$$

In this case, the isotropy subalgebra around the base point $(0, 0, 1, 0)$ is generated by $\{e_1 + 2e_6, e_4\}$, so the factor space is four dimensional.

Above arguments give the following result for the homogeneous Siklos space-times.

Theorem 3.1. *A Siklos space-time (M, g), where g is defined in the local coordinates (x_1, x_2, x_3, x_4) as (1), is homogeneous if the defining function H is $H = \pm x_3^{\alpha}$. In this case, the Lie algebra of local isometries is generated by the non-zero commutators either of (6) for $\alpha \neq -2$ or (7) for $\alpha = -2$.*

4. Algebraic description of homogeneous Siklos space-times

Following identification of the homogeneous Siklos space-times in the previous section, we focus on the study of these spaces from an algebraic point of view.

Case(I): Let $H = \pm x_3^{\alpha}$, $\alpha \neq -2$, in this case the Lie algebra $\mathfrak{g}$ of local isometries is described by (6) and the isotropy subalgebra is generated by $\{e_4\}$. We set $h_1 = e_4, u_1 = e_1, u_2 = e_2, u_3 = e_3, u_4 = e_5$, then $\mathfrak{h} = \mathrm{span}\{h_1\}$ and the factor subspace $\mathfrak{m} = \mathrm{span}\{u_1, u_2, u_3, u_4\}$, where the Lie algebra $\mathfrak{g}$ is

$$\begin{array}{llll} [h_1, u_2] = u_3, & [h_1, u_3] = -u_1, & [h_1, e_4] = \alpha h_1, & \\ [u_1, u_4] = (2+\alpha)u_1, & [u_2, u_4] = (2-\alpha)u_2, & [u_3, u_4] = 2u_3, & \alpha \neq -2. \end{array}$$

The isotropy representation for h_1 is deduced by (2) as

$$H_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, a bilinear form g is invariant (so is an invariant metric) if and only if ${}^tH_1 g + gH_1 = 0$, which immediately gives $g_{11} = g_{13} = g_{14} = g_{23} = g_{34} = g_{33} - g_{12} = 0$, i.e., the invariant metric g is deduced as

$$g = \begin{pmatrix} 0 & a & 0 & 0 \\ a & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}. \tag{8}$$

This metric is non-degenerate whenever $ad \neq 0$. We set $\nabla_i := \nabla(u_i), i = 1, \cdots, 4$ and using (3), components of the Levi-Civita connection are calculated as

$$\nabla_1 = \begin{pmatrix} 0 & \frac{2c}{d} & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{2a}{d} & 0 & 0 \end{pmatrix}, \quad \nabla_2 = \begin{pmatrix} \frac{2c}{d} & -\frac{cb(\alpha-2)}{ad} & 0 & \frac{2c^2-bd\alpha}{ad} \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ -\frac{2a}{d} & \frac{b(\alpha-2)}{d} & 0 & -\frac{2c}{d} \end{pmatrix},$$

$$\nabla_3 = \begin{pmatrix} 0 & 0 & \frac{2c}{d} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -\frac{2a}{d} & 0 \end{pmatrix}, \quad \nabla_4 = \begin{pmatrix} -\alpha & \frac{2c^2-bd\alpha}{ad} & 0 & \frac{c(2-\alpha)}{a} \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{2c}{d} & 0 & 0 \end{pmatrix}.$$

Then, (4) yields to the following components of the curvature tensor $R_{ij} := R(u_i, u_j), i < j = 1, \cdots, 4$ as

$$R_{12} = \begin{pmatrix} -\frac{4a}{d} & -\frac{4b}{d} & 0 & -\frac{4c}{d} \\ 0 & \frac{4a}{d} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_{13} = \begin{pmatrix} 0 & 0 & -\frac{4a}{d} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{4a}{d} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{14} = \begin{pmatrix} 0 & -\frac{4c}{d} & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{4a}{d} & 0 & 0 \end{pmatrix}, \qquad R_{23} = \begin{pmatrix} 0 & 0 & \frac{2b\alpha}{d} & 0 \\ 0 & 0 & -\frac{4a}{d} & 0 \\ \frac{4a}{d} & \frac{2b(2-\alpha)}{d} & 0 & \frac{4c}{d} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$R_{24} = \begin{pmatrix} 0 & \frac{2cb\alpha(2-\alpha)}{ad} & 0 & \frac{2b\alpha(2-\alpha)}{a} \\ 0 & -\frac{4c}{d} & 0 & -4 \\ 0 & 0 & 0 & 0 \\ \frac{4a}{d} & \frac{2b(-2\alpha+\alpha^2+2)}{d} & 0 & \frac{4c}{d} \end{pmatrix}, \qquad R_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{4c}{d} & 0 & -4 \\ 0 & 0 & \frac{4a}{d} & 0 \end{pmatrix}.$$

We calculate the Ricci tensor as follows

$$\varrho = \begin{pmatrix} 0 & -\frac{12a}{d} & 0 & 0 \\ -\frac{12a}{d} & -\frac{2b\left(6-3\alpha+\alpha^2\right)}{d} & 0 & -\frac{12c}{d} \\ 0 & 0 & -\frac{12a}{d} & 0 \\ 0 & -\frac{12c}{d} & 0 & -12 \end{pmatrix}. \tag{10}$$

The Ricci operator is of degenerate Segre type $[(11, 2)]$ in this case.

Case(II): Let $H = \pm x_3^{-2}$, the Lie algebra $\mathfrak{g}$ of local isometries is described by (7) and the isotropy subalgebra is spanned by $\{e_1 + 2e_6, e_4\}$. We set $h_1 = e_4, h_2 = e_2 + 2e_6, u_1 = e_2, u_2 = e_3, u_3 = e_5, u_4 = e_6$, then $\mathfrak{h} = \operatorname{span}\{h_1, h_2\}$ and $\mathfrak{m} = \operatorname{span}\{u_1, u_2, u_3, u_4\}$. In this new basis, the Lie algebra $\mathfrak{g}$ is specified by the following non-zero commutators

$$\begin{array}{llll} [h_1, u_1] = u_2, & [h_1, u_2] = -h_2 + 2u_4, & [h_1, u_3] = -2h_1, & [h_2, u_1] = -u_3, \\ [h_2, u_2] = 2h_1, & [h_2, u_3] = -8u_4, & [u_1, u_3] = 4u_1, & [u_1, u_4] = \frac{1}{2}u_3, \\ [u_2, u_3] = 2u_2, & [u_2, u_4] = -h_1, & [u_3, u_4] = 4u_4. & \end{array}$$

The isotropy representation for the generators h_1 and h_2 of $\mathfrak{h}$ are determined by the following matrices respectively

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 \end{pmatrix}.$$

By direct calculations, a bilinear form g is invariant if and only if

$$g = \begin{pmatrix} a & 0 & 0 & b \\ 0 & -2b & 0 & 0 \\ 0 & 0 & -8b & 0 \\ b & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

which is clearly non-degenerate whenever $b \neq 0$. Non-zero components of the Levi-Civita connection are now deduced as

$$\nabla_1 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{2b} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{2a}{b} & 0 \end{pmatrix}, \quad \nabla_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\nabla_3 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2a}{b} & 0 & 0 & 2 \end{pmatrix}, \quad \nabla_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

Then, the curvature tensor is deduced by the following components

$$R_{12} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\frac{a}{b} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{a}{b} & 0 & 0 \end{pmatrix}, \quad R_{13} = \begin{pmatrix} 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{5a}{2b} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{16a}{b} & 0 \end{pmatrix},$$
$$R_{14} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a}{2b} & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad R_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$
$$R_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad R_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}.$$

Now, one can calculate the Ricci tensor as

$$\varrho = \begin{pmatrix} \frac{4a}{b} & 0 & 0 & \frac{3}{2} \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -12 & 0 \\ \frac{3}{2} & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

The Ricci operator is again of the Segre type $[(11,2)]$ in this case.

Theorem 4.1. *A homogeneous Siklos space-time $(M = G/H, g)$ of class (I) is*

- *neither flat nor Ricci flat.*
- *Locally symmetric if and only if $\alpha b = 0$.*
- *Ricci parallel if and only if $\alpha(\alpha - 3)b = 0$.*
- *Einstein if and only if be Ricci parallel.*
- *Conformally flat if and only if $\alpha(\alpha - 1)b = 0$.*

Proof. According to (9) and (10), it is evident that $(G/H, g)$ is never (Ricci) flat. Then, for the $(0, 4)$ curvature tensor field R, standard calculations give the non-zero components of the covariant derivative of the curvature tensor up to symmetries as

$$\begin{aligned} &(\nabla(u_2)R)_{1224} = \tfrac{4ab\alpha(\alpha-2)}{d}, \quad (\nabla(u_2)R)_{2334} = \tfrac{4ab\alpha}{d}, \quad (\nabla(u_2)R)_{2442} = \tfrac{8bc\alpha(\alpha-2)}{d}, \\ &(\nabla(u_3)R)_{2342} = \tfrac{4ab\alpha(\alpha-1)}{d}, \\ &(\nabla(u_4)R)_{2332} = \tfrac{4ab\alpha^2}{d}, \quad (\nabla(u_3)R)_{2422} = 4\alpha^2 b(\alpha - 2). \end{aligned}$$

Since $ad \neq 0$, the above relations immediately imply to the second statement.

By (10), non-zero components of the covariant derivative of the Ricci tensor up to symmetries are

$$(\nabla(u_2)\varrho)_{24} = \frac{4b\alpha(\alpha-3)}{d}, \qquad (\nabla(u_4)\varrho)_{22} = \frac{4b\alpha^2(\alpha-3)}{d},$$

which gives the third statement.

Applying (8) and (10), the Einstein equation $\varrho = \lambda g$ is valid for some real constant λ when ever $\lambda d + 12 = 0$ and $b(\alpha^2 - 3\alpha) = 0$. This concludes the fourth statement.

To study the conformally flat condition, we calculate the components of the Weyl conformal tensor field W using the equation

$$W_{ijhk} = R_{ijhk} - \frac{1}{2}(g_{ih}\varrho_{jk} - g_{jh}\varrho_{ik} - g_{ik}\varrho_{jh} + g_{jk}\varrho_{ih}) + \frac{\tau}{6}(g_{ih}g_{jk} - g_{jh}g_{ik}).$$

Direct calculations yield the non-zero components of the Weyl conformal tensor field up to symmetries are

$$W_{2323} = \tfrac{ab\alpha(\alpha-1)}{d}, \quad W_{2442} = b\alpha(\alpha - 1).$$

This shows the weyl tensor will vanish and so $(G/H, g)$ is conformally flat when ever $\alpha(\alpha - 1)b = 0$. Thus, the proof is complete. □

Remark 4.2. *Through of Einstein examples of the homogeneous Siklos space-times of class (I), the case* $\alpha = 3$ *corresponds to Kaigorodov space-time, which is the only homogeneous type-N solution of the Einstein vacuum field equations with* $\Lambda \neq 0$ *[18, 25].*

Theorem 4.3. *A homogeneous Siklos space-time* $(M = G/H, g)$ *of class (II) is*

- *neither flat nor Ricci flat.*
- *Locally symmetric and Ricci parallel if and only if* $a = 0$.
- *Einstein if and only if be Ricci parallel.*
- *Conformally flat if and only if* $a = 0$.

Proof. By (12) and (13), clearly $(G/H, g)$ is never (Ricci) flat. Then, for the $(0, 4)$ curvature tensor field R, standard calculations give the non-zero components of the covariant derivative of the curvature tensor up to symmetries as

$$\begin{aligned} &(\nabla(u_1)R)_{1232} = 2a, \quad (\nabla(u_1)R)_{1314} = 4a, \\ &(\nabla(u_2)R)_{1231} = 6a, \\ &(\nabla(u_3)R)_{1221} = 4a, \end{aligned}$$

which shows that space is locally symmetric if and only if $a = 0$. By similar arguments using (13), non-zero components of the covariant derivative of the Ricci tensor up to symmetries are

$$(\nabla(u_1)\varrho)_{13} = -\frac{5a}{b}, \qquad (\nabla(u_3)\varrho)_{11} = \frac{10a}{b},$$

that gives the Ricci parallel condition when ever $a = 0$. This shows validity of the second statement.

Applying (11) and (13), the Einstein equation $\varrho = \lambda g$ establishes for some real constant λ, when ever $2\lambda b - 3 = 0$ and $a = 0$.

Now, non-Zero components of the Weyl conformal tensor field up to symmetries are

$$W_{1212} = \tfrac{3}{2}a \quad W_{1331} = 6a,$$

which shows that $(G/H, g)$ is conformally flat if and only if $a = 0$. This concludes the fourth statement and the proof is complete. □

Remark 4.4. *Class (II) of Siklos space-times corresponds to the pure radiation solution of Petrov type-N with a* G_6 *isometry group, first described by Defrise [14].*

5. Generalized Ricci solitons on homogenous Siklos space-times

To study generalized Ricci solitons on homogeneous Siklos space-times, we study the equation (5) on the homogeneous cases which were described in Section 4.

Case (I): Let (M, g) be a homogeneous Siklos space-time of class I. With respect to the basis $\{u_1, \cdots, u_4\}$ for the factor space $\mathfrak{m}$, let $V = v_1u_1 + \cdots, v_4u_4$ be and arbitrary vector field, where $v_1, \cdots, v_4$ are arbitrary real coefficients. Applying the invariant metric (8), one has

$$V^\flat = av_2\theta^1 + (cv_4 + bv_2 + av_1)\theta^2 + av_3\theta^3 + (dv_4 + cv_2)\theta^4,$$

where $\{\theta^1, \cdots, \theta^4\}$ is the dual basis of $\{u_1, \cdots, u_4\}$. Straight forward calculations using the relation $(\mathcal{L}_V g)(e_i, e_j) = \langle \nabla(e_i)V, e_j\rangle + \langle e_i, \nabla(e_j)V\rangle$ yields

$$\mathcal{L}_V g = \begin{pmatrix} 0 & 4v_4a & 0 & v_2a(\alpha-2) \\ 4v_4a & 2v_4b(2-\alpha) & 0 & \varphi \\ 0 & 0 & 4v_4a & -2av_3 \\ v_2a(\alpha-2) & \varphi & -2av_3 & 2v_2c(\alpha-2) \end{pmatrix}, \tag{14}$$

where $\varphi = av_1(-2-\alpha) + (bv_2 - cv_4)(\alpha - 2)$. Now, using (8), (10) and (14), the vector filed V satisfies (5) if and only if the following system of algebraic equations establish

$$\begin{cases} \eta a^2 v_2^2 = 0, \\ \eta a^2 v_2 v_3 = 0, \\ \eta(cv_4 + bv_2 + av_1)av_3 = 0, \\ \eta(dv_4 + cv_2)(av_3 - 1) = 0, \\ v_2 a((\alpha - 2) + 2\eta(dv_4 + cv_2)) = 0, \\ 24\beta + 2cv_2(\alpha - 2) + 2\eta(dv_4 + cv_2)^2 - 2d\lambda = 0, \\ 24\beta\frac{a}{d} + 4av_4 + 2\eta a^2 v_3^2 - 2\lambda a = 0, \\ 24\beta\frac{a}{d} + 4av_4 + 2\eta a v_2(cv_4 + bv_2 + av_1) - 2\lambda a = 0, \\ 4\beta(6 - 3\alpha + \alpha^2)\frac{b}{d} - 2v_4 b(\alpha - 2) + 2\eta(cv_4 + bv_2 + av_1)^2 - 2\lambda b = 0, \\ 24\beta\frac{c}{d} - av_1(2+\alpha) + (bv_2 - cv_4)(\alpha - 2) \\ \qquad +2\eta(cv_4 + bv_2 + av_1)(dv_4 + cv_2) - 2\lambda c = 0. \end{cases} \tag{15}$$

Case (II): In this case, by direct calculations using the Equation (11) have

$$V^\flat = (av_1 + bv_4)\theta^1 - 2bv_2\theta^2 - 8bv_3\theta^3 + bv_1\theta^4,$$

and also,

$$\mathcal{L}_V g = \begin{pmatrix} 8av_3 & 0 & -4av_1 & 0 \\ 0 & -8bv_3 & 4bv_2 & 0 \\ -4av_1 & 4bv_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

Now, using (11), (13) and (16), the equation (5) establishes if and only if the following system of algebraic equations establish

$$\begin{cases} \eta b^2 v_1^2 = 0, \\ \eta b^2 v_1 v_2 = 0, \\ \eta b^2 v_1 v_3 = 0, \\ \eta(bv_4 + av_1)bv_2 = 0, \\ bv_2 + 8\eta b^2 v_2 v_3 = 0, \\ (a + 4\eta abv_3)v_1 + 4\eta b^2 v_3 v_4 = 0, \\ 3\beta + 16\eta b^2 v_3^2 + 2\lambda b = 0, \\ 3\beta + 4\eta b^2 v_2^2 + (2\lambda - 4v_3)b = 0, \\ 3\beta - 2\eta b^2 v_1 v_4 - (2\eta v_1^2 a - 2\lambda)b = 0, \\ 4\beta\frac{a}{b} - 4v_3 a - \eta(bv_4 + av_1)^2 + \lambda a = 0. \end{cases} \tag{17}$$

To classify generalized Ricci solitons on the homogeneous Siklos space-times of Type (I) and (II), it is enough to solve the corresponding system of algebraic equations.

Theorem 5.1. *A homogeneous Siklos space-time $M = G/H$ equipped with the invariant metric g is a (non-Einstein) generalized Ricci soliton with the vector field $V = v_1u_1 + \cdots + v_4u_4$, if and only if one of the following cases occurs*

- **Case (I):** *g is the invariant metric (8) and*
  - *i)* $\beta = \frac{1}{\eta(\alpha-3)}, d = \frac{4\alpha}{\eta\lambda(\alpha-3)}, v_1 = \frac{c\lambda(3-\alpha)}{2\alpha a}, v_2 = v_3 = 0, v_4 = \frac{\lambda(\alpha-3)}{2\alpha}$
  - *ii)* $\alpha = 2, \beta = \eta = \lambda = 0, v_1 = v_3 = v_4 = 0.$
- **Case (II):** *g is the invariant metric (11) and*

i) $\beta = \frac{2\eta b^3 v_4^2}{5a}, \lambda = -\frac{3\eta b^2 v_4^2}{5a}, v_1 = v_2 = v_3 = 0,$

ii) $\beta = \eta = \lambda = 0, v_1 = v_2 = v_3 = 0.$

Proof. According to the arguments before, we solve the system of Equations (15) and (17). We bring the details for the Case (II) and the first case would be handled by similar arguments. To discard the Einstein solutions, by the Theorem 4.3 we suppose $a \neq 0$. On the other hand, $b \neq 0$ for non-degeneracy of the metric tensor. From the first equation in (17), we have $\eta = 0$ or $v_1 = 0$.

Let $\eta = 0$, then from the fifth and sixth equations have $v_1 = v_2 = 0$. In this case, the seventh equation reads $3\beta + 2\lambda b = 0$ which gives in the eighth equation $v_3 = 0$. Now the last equation gives $a(4\beta + \lambda b) = 0$ which since $a \neq 0$, with the seventh equation concludes $\beta = \lambda = 0$. So, the case ii) in the second statement is deduced. Clearly this case shows that $(G/H, g)$ is a Ricci soliton with the invariant vector field $V = u_4$.

Let $v_1 = 0$ and $\eta \neq 0$, then the ninth equation gives $3\beta + 2\lambda b = 0$ and from the seventh equation have $v_3 = 0$ and immediately the fifth equation reads $v_2 = 0$. Now, if set $\beta = -\frac{2}{3}\lambda b$ then the last equation gives $\frac{5}{3}\lambda a + \eta b^2 v_4^2 = 0$ which concludes $\lambda = -\frac{3\eta b^2 v_4^2}{5a}$. This shows validity of the second statement and completes the proof. □

Remark 5.2. *Theorem 5.1 shows that for the class (I) of homogeneous Siklos space-times, Ricci solitons exist just for $\alpha = 2$, with the vector field $V = u_2$, while homogeneous Siklos space-times of class (II) are always a Ricci soliton with the vector field $V = u_4$.*

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References

[1] A. Arvanitoyeorgos, *An introduction to Lie groups and the geometry of homogeneous spaces*, vol. 22 of Student Mathematical Library, American Mathematical Society, Providence, RI, 2003. Translated from the 1999 Greek original and revised by the author.

[2] J. Bičák and J. Podolský, *Gravitational waves in vacuum spacetimes with cosmological constant. I. Classification and geometrical properties of nontwisting type N solutions*, J. Math. Phys., 40 (1999), pp. 4495–4505.

[3] ——, *Gravitational waves in vacuum spacetimes with cosmological constant. II. Deviation of geodesics and interpretation of nontwisting type N solutions*, J. Math. Phys., 40 (1999), pp. 4506–4517.

[4] D. M. J. Calderbank and H. Pedersen, *Einstein-Weyl geometry*, in Surveys in differential geometry: essays on Einstein manifolds, vol. 6 of Surv. Differ. Geom., Int. Press, Boston, MA, 1999, pp. 387–423.

[5] G. Calvaruso, *Siklos spacetimes as homogeneous Ricci solitons*, Classical Quantum Gravity, 36 (2019), pp. 095011, 13.

[6] ——, *Conformally flat Siklos metrics are Ricci solitons*, Axioms, 9 (2020), pp. 1–13.

[7] ——, *The Ricci soliton equation for homogeneous Siklos spacetimes*, Note Mat., 41 (2021), pp. 31–44.

[8] G. Calvaruso and A. Fino, *Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces*, Canad. J. Math., 64 (2012), pp. 778–804.

[9] G. Calvaruso, M. Kaflou, and A. Zaeim, *On the symmetries of Siklos spacetimes*, Gen. Relativity Gravitation, 54 (2022), pp. Paper No. 60, 26.

[10] G. Calvaruso and A. Zaeim, *The Bach-flat and conformally Einstein equations for Siklos spacetimes*. Submitted.

[11] ——, *A complete classification of Ricci and Yamabe solitons of non-reductive homogeneous 4-spaces*, J. Geom. Phys., 80 (2014), pp. 15–25.

[12] H.-D. Cao, *Recent progress on Ricci solitons*, in Recent advances in geometric analysis, vol. 11 of Adv. Lect. Math. (ALM), Int. Press, Somerville, MA, 2010, pp. 1–38.

[13] P. T. Chruściel, H. S. Reall, and P. Tod, *On non-existence of static vacuum black holes with degenerate components of the event horizon*, Classical Quantum Gravity, 23 (2006), pp. 549–554.

[14] L. Defrise, *Groupes d'isotropie et groupes de stabilité conforme dans les espaces Lorentziens*, PhD thesis, Université Libre de Bruxelles.

[15] J. Ehlers and W. Kundt, *Exact solutions of the gravitational field equations*, in Gravitation: An introduction to current research, Wiley, New York-London, 1962, pp. 49–101.

[16] A. Ghosh, *Ricci soliton and Ricci almost soliton within the framework of Kenmotsu manifold*, Carpathian Math. Publ., 11 (2019), pp. 56–69.

[17] R. Karami, A. Zaeim, and A. Haji-Badali, *Ricci solitons and geometry of four dimensional Einstein-like neutral Lie groups*, Period. Math. Hungar., 78 (2019), pp. 58–78.

[18] V. R. Kaĭgorodov, *Einstein spaces of maximum mobility*, Dokl. Akad. Nauk SSSR, 146, pp. 793–796.

[19] J. Lauret, *Ricci soliton homogeneous nilmanifolds*, Math. Ann., 319 (2001), pp. 715–733.

[20] ——, *Ricci soliton solvmanifolds*, J. Reine Angew. Math., 650 (2011), pp. 1–21.

[21] M. Mohseni, *Vacuum polarization in Siklos spacetimes*, Phys. Rev. D, 97 (2018), pp. 024006, 6.

[22] P. Nurowski and M. Randall, *Generalized Ricci solitons*, J. Geom. Anal., 26 (2016), pp. 1280–1345.

[23] I. Ozsváth, I. Robinson, and K. Rózga, *Plane-fronted gravitational and electromagnetic waves in spaces with cosmological constant*, J. Math. Phys., 26 (1985), pp. 1755–1761.

[24] S. Pahan, T. Dutta, and A. Bhattacharyya, *Ricci soliton and η-Ricci soliton on generalized Sasakian space form*, Filomat, 31 (2017), pp. 4051–4062.

[25] J. Podolský, *Interpretation of the Siklos solutions as exact gravitational waves in the anti-de Sitter universe*, Classical Quantum Gravity, 15 (1998), pp. 719–733.

[26] J. Podolský, *Exact non-singular waves in the anti-de Sitter universe*, Gen. Relativity Gravitation, 33 (2001), pp. 1093–1113.

[27] J. Podolský and J. B. Griffiths, *Impulsive waves in de sitter and anti-de sitter spacetimes generated by null particles with an arbitrary multipole structure*, Class. Quantum Gravity, 15 (1998), p. 453.

[28] M. Randall, *Local obstructions to projective surfaces admitting skew-symmetric Ricci tensor*, J. Geom. Phys., 76 (2014), pp. 192–199.

[29] S. T. C. Siklos, *Lobatchevski plane gravitational waves*, in Galaxies, Axisymmetric Systems and Relativity, M. A. H. MacCallum, ed., Cambridge University Press, 1985.

[30] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, second ed., 2003.

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