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Original Article

Some results concerning asymptotic distribution of functional linear regression with points of impact

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ABSTRACT: Lately, issues related to functional linear regression models with points of impact have garnered significant interest. While the literature has addressed the estimation of parameters for this model with scalar response, less attention has been paid to the asymptotic distribution of the impact points coefficients estimators. In recent literature, the asymptotic distribution has been pointed out in a particular case, but the demonstration of its validity has not been adequately addressed. By explicating the necessary requirements, we derive an important part of the asymptotic distribution of the impact points coefficients estimators in a general setting. This is a fundamental result for finding the asymptotic distribution of the impact points coefficients estimators. Moreover, we perform a simulation study to exhibit the efficiency of the obtained results.

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1. Introduction

The functional linear regression (FLR) model has been gaining prominence in recent years due to its growing applicability in various scientific fields, such as economics, biology, engineering, and social sciences. In a standard FLR, a real-valued scalar response is related to a functional predictor, allowing for the analysis of complex data structures and multivariate patterns that capture the interactions among variables. The need for novel statistical methods that provide better insights into the underlying interrelations of such functional data is ever-growing. Several estimation methods have been developed for FLR, including the popular functional principal component analysis (FPCA) method. For instance, consider [2, 3, 4], and [8]. Also some other works employing the spline approach are the work of [1] and [6].

In FLR, only the global effect of the predictor on the response variable can be evaluated and we cannot assess the effect of the predictor's local characteristics on the response. For this purpose, e.g. [11] estimated the slope

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function, considering instances where its values are equal to zero in specific subregions. In recent years, a growing body of research has focused on the role of points of impact in FLR. Points of impact are specific locations in the domain of the functional predictor where the effect of the predictor is most influential. Among works in this area, [7] explored the determination and assessment of a point of impact within the time domain in FLR, while they did not consider the global effect of the predictor on the response variable. [5] considered the functional linear regression with points of impact (FLRPI) that contains the effects of predictor evaluated at multiple points of impact in addition to the global effect of the predictor on the response variable. [6] employed the spline technique to estimate the slope function and coefficients in FLRPI. [9] obtained some asymptotic properties of FLRPI.

We discuss FLRPI model, introduced by [5]. Though researchers like [5] and [6] have studied the parameters estimation of this model, the asymptotic distribution has not received as much focus in the existing literature. Without considering the global effect of the predictor on the response variable, [5] derived the asymptotic distribution of the impact points coefficients estimators, while neglecting to provide a proof. In this paper, we derive an important part of the asymptotic distribution of the impact points coefficients estimators in a general setting. The paper is structured as follows: In Section 2, we present the studied FLRPI model. Section 3 includes the necessary assumptions and the target asymptotic distribution. In Section 4, we perform a simulation study utilizing the Ornstein-Uhlenbeck process to illustrate the performance of the obtained results. Finally, we have dedicated section 5 to the conclusion.

2. Model and notations

We study a model that consists of a scalar response variable Y and a functional predictor variable $X \in L^2(I)$, where $I = [a, b]$ is a bounded interval of $\mathbb{R}$ and $L^2(I)$ is the space of all square-integrable functions on I. We assume that the dataset is composed of independent observations (X_i, Y_i), $i = 1, \dots, n$, of the pair (X, Y). The functional variable X defined on I, is such that $E[\int_I X^2(t)dt] < \infty$. Without loss of generality, we can assume that the variables have been centered, that is, $E(Y) = 0$ and $E[X(t)] = 0$ for $t \in [a, b]$. Subsequently, the Y_i's are produced using the following FLRPI model

$$Y_i = \int_I \beta(t) X_i(t)\, dt + \sum_{r=1}^{S} \beta_r X_i(\tau_r) + \varepsilon_i, \quad i = 1, \dots, n. \tag{1}$$

The error terms ε_i are i.i.d random variables with zero mean and finite variance σ^2, the slope function β is an unknown, bounded square-integrable function on the interval I, and the errors ε_i are also independent from the X_i.

In Model (1), the global effect of the trajectory $X_i(\cdot)$ on Y_i is represented by the term $\int_I \beta(t) X_i(t)\, dt$. The model also considers an unknown number $S \in \mathbb{N}$, denoted by points of impact $\tau_1, \dots, \tau_S$, that the response variable Y_i is substantially affected by the corresponding functional values $X_i(\tau_1), \dots, X_i(\tau_S)$ at these time points. Model (1) includes the unknown parameters $\beta(t)$, the number $S \geq 0$, τ_r and β_r, $r = 1, \dots, S$, which must be estimated from the given data. A necessary condition for the identifiability of the parameters of Model (1) is the presence of a "specific local variation" feature. This implies that at least some part of the local fluctuations of the predictor process $X_i(t)$ within the close vicinity of any point of impact is fundamentally uncorrelated with the trajectories outside this limited area. Among the stochastic processes that possess this feature are Brownian motion, fractional Brownian motion, and the Ornstein–Uhlenbeck process (for more details on the specific local variation feature, see [5]). We assume that every impact point is located in the interior of the interval, $\tau_r \in (a, b)$, $r = 1, \dots, S$. We also assume that X has a continuous covariance function $\Gamma(t, s)$ for $t, s \in I$, and that the eigenvalues $\lambda_1, \lambda_2, \dots$ and eigenfunctions $\psi_1, \psi_2, \dots$ of the covariance operator with kernel $\Gamma(\cdot, \cdot)$ constitute a strictly decreasing sequence and an orthonormal basis for the $L^2(I)$, respectively. In a similar manner, the eigenvalues and eigenfunctions of the sample covariance operator with kernel $\hat{\Gamma}(\cdot, \cdot)$ are represented by $\hat{\lambda}_j$ and $\hat{\psi}_j$, respectively.

In the present study, we employ the technique proposed by [5] to estimate the number and locations of impact points. Subsequently, we estimate the slope function along with the regression coefficients by utilizing FPCA and the least squares approaches, respectively. Let $\theta_{ij} = \langle X_i, \psi_j \rangle$, $\hat{\theta}_{ij} = \langle X_i, \hat{\psi}_j \rangle$ and $\alpha_j = \langle \beta, \psi_j \rangle$ for all i, j. The Karhunen-Loève expansions for the functions X_i and β can be expressed as $\sum_{j=1}^{\infty} \theta_{ij} \psi_j$ and $\sum_{j=1}^{\infty} \alpha_j \psi_j$, respectively. It can then be demonstrated that $\int_a^b \beta(t) X_i(t) dt = \sum_{j=1}^{\infty} \alpha_j \theta_{ij}$. With estimates $\widehat{S}$, $\hat{\tau}_1, \dots, \hat{\tau}_{\widehat{S}}$, and an appropriate cut-off parameter k, we obtain

$$Y_i = \sum_{j=1}^{k} \alpha_j \hat{\theta}_{ij} + \sum_{r=1}^{\hat{S}} \beta_r X_i(\hat{\tau}_r) + \varepsilon_i^*, \quad i = 1, \dots, n,$$

where

$$\varepsilon_i^* = \sum_{j=1}^{\infty} \alpha_j \theta_{ij} - \sum_{j=1}^{k} \alpha_j \hat{\theta}_{ij} + \sum_{r=1}^{S} \beta_r X_i(\tau_r) - \sum_{r=1}^{\hat{S}} \beta_r X_i(\hat{\tau}_r) + \varepsilon_i, \quad i = 1, \ldots, n.$$

The residual sum of squares $\sum_{i=1}^{n}(Y_i - \sum_{j=1}^{k} a_j \hat{\theta}_{ij} - \sum_{r=1}^{\hat{S}} b_r X_i(\hat{\tau}_r))^2$, can be minimized over all a_j, b_r for $j = 1, \ldots, k$ and $r = 1, \ldots, \widehat{S}$ to obtain estimates $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_{\hat{S}})^T$ for $\beta = (\beta_1, \ldots, \beta_S)^T$, and $\hat{\alpha}_1, \ldots, \hat{\alpha}_k$ for $\alpha_1, \ldots, \alpha_k$. The slope function estimator $\hat{\beta}(t)$, is calculated as $\hat{\beta}(t) = \sum_{j=1}^{k} \hat{\alpha}_j \hat{\psi}_j(t)$.

3. Asymptotic distribution

This section includes the necessary assumptions and the obtained asymptotic distribution for Model (1).

Assumption 1. *The process X is Gaussian and the error terms $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d normal random variables.*

Define $\Gamma^{[k]}(t,s) = \sum_{j=k+1}^{\infty} \lambda_j \psi_j(t)\psi_j(s)$. $\boldsymbol{M}_k$ represents the $S \times S$ matrix with elements $\Gamma^{[k]}(\tau_r, \tau_s)$ and the minimum eigenvalue of the matrix $\boldsymbol{M}_k$ is denoted by $\lambda_{min}(\boldsymbol{M}_k)$.

Assumption 2. *(a) There exist constants $\mu > 1$ and C_0, where $\sigma^2 < C_0 < \infty$, such that for all $j \geq 1$, $\lambda_j \leq C_0 j^{-\mu}$ and $\lambda_j - \lambda_{j+1} \geq C_0^{-1} j^{-\mu-1}$.*

(b) For all t, $\beta(t) = \sum_{j=1}^{\infty} \alpha_j \psi_j(t)$, and there exists constants $0 < C_1 < \infty$ and $\nu > 1 + \frac{1}{2}\mu$ such that $|\alpha_j| \leq C_1 j^{-\nu}$.

(c) $\sup_t \sup_j \psi_j(t)^2 \leq C_\psi$ for some $C_\psi < \infty$.

(d) There exists a constant $0 < D < \infty$ such that $\lambda_{\min}(\boldsymbol{M}_k) \geq D k^{-\mu+1}$ for every k.

Given $\hat{S} = S$, it can be proved that

$$\widehat{\boldsymbol{\beta}} = \Big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]} (\boldsymbol{X}_i^{[k]})^T\Big)^{-1} \Big[\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]} \big(\sum_{r=1}^{S} \beta_r X_i^{[k]}(\tau_r) + \sum_{j=k+1}^{n} \tilde{\alpha}_j \hat{\theta}_{ij} + \varepsilon_i\big)\Big],$$

where $\tilde{\alpha}_j = \langle \beta, \hat{\psi}_j \rangle$, $\boldsymbol{X}_i^{[k]} = [X_i^{[k]}(\hat{\tau}_1), \ldots, X_i^{[k]}(\hat{\tau}_S)]^T$, and $X_i^{[k]}(t) = X_i(t) - \sum_{j=1}^{k} \hat{\theta}_{ij} \widehat{\psi}_j(t)$. [5] pointed out that $\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} N(\boldsymbol{0}, \sigma^2 \boldsymbol{\Sigma}_\tau^{-1})$, where $\boldsymbol{\Sigma}_{\boldsymbol{\tau}} = E[\boldsymbol{X}_i(\boldsymbol{\tau})\boldsymbol{X}_i(\boldsymbol{\tau})^T]$ and $\boldsymbol{X}_i(\boldsymbol{\tau}) = [X_i(\tau_1), \ldots, X_i(\tau_S)]^T$, while neglecting to provide a proof. This conclusion was derived under the additional assumption that the integral term in Model (1) is zero, which is not generally true. In this research, the general case where the integral term in Model (1) is not zero is considered. As a crucial step toward determining the asymptotic distribution of $\widehat{\boldsymbol{\beta}}$, we will show in Theorem 3.1 that the asymptotic distribution of $\big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]} (\boldsymbol{X}_i^{[k]})^T\big)^{-1} \times \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \boldsymbol{X}_i^{[k]} \varepsilon_i$ is normal.

Theorem 3.1. *If Assumptions 1 and 2 hold, $k = O(n^{\frac{1}{\mu+2\nu}})$, $n^{\frac{1}{\mu+2\nu}} = O(k)$, then under the conditions of Theorem 4 of [5], we have*

$$Z_n = \big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]} (\boldsymbol{X}_i^{[k]})^T\big)^{-1} \times \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \boldsymbol{X}_i^{[k]} \varepsilon_i \xrightarrow{\;\;d\;\;} N(\boldsymbol{0}, \sigma^2 \boldsymbol{M}_k^{-1}). \tag{2}$$

Proof. Given $\widehat{S} = S$, let $\boldsymbol{x}_{n,i} = \boldsymbol{a}_{n,i}\varepsilon_i$ with $\boldsymbol{a}_{n,i} = \frac{1}{\sqrt{n}} \boldsymbol{X}_i^{[k]}$ where $1 \leq i, k \leq n$. It is clear that the $\boldsymbol{x}_{n,i}$s are independent for each n. By Assumption 1, we can conclude that $E[\boldsymbol{x}_{n,i}] = \boldsymbol{0}$ and $Cov[\boldsymbol{x}_{n,i}] = \frac{\sigma^2}{n} E\big[\boldsymbol{X}_i^{[k]} (\boldsymbol{X}_i^{[k]})^T\big]$. Under the conditions of Theorem 4 of [5], if Assumptions 1 and 2 hold, $k = O(n^{\frac{1}{\mu+2\nu}})$, $n^{\frac{1}{\mu+2\nu}} = O(k)$, then

$$\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]} (\boldsymbol{X}_i^{[k]})^T = \boldsymbol{M}_k + o_p\big(n^{\frac{-\mu+1}{\mu+2\nu}}\big), \tag{3}$$

(see (A.15) of [5]). Equation (3) and Assumption 2(a) yield

$$\lim_{n\to\infty} \sum_{i=1}^{n} Cov[\boldsymbol{x}_{n,i}] = \lim_{n\to\infty} \frac{\sigma^2}{n} E\big[\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]} (\boldsymbol{X}_i^{[k]})^T\big] = \sigma^2 \boldsymbol{M}_k. \tag{4}$$

Let $A = [\boldsymbol{a}_{n,1}, \dots, \boldsymbol{a}_{n,n}]^T$. We can write

$$\sum_{i=1}^{n} \|\boldsymbol{a}_{n,i}\|^2 = trace(AA^T) = trace(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T). \tag{5}$$

Equations (3) and (5) imply that, for every $\epsilon > 0$,

$$\begin{aligned}\sum_{i=1}^{n} E\big[\|\boldsymbol{a}_{n,i}\varepsilon_i\|^2 \boldsymbol{I}\{\|\boldsymbol{a}_{n,i}\varepsilon_i\| > \epsilon\}\big] &\leq E\Bigg[\bigg(\max_{1\leq i\leq n} E\big[\varepsilon_i^2 \boldsymbol{I}\{\|\boldsymbol{a}_{n,i}\||\varepsilon_i|> \epsilon\}|\mathcal{X}\big]\bigg)\bigg(trace(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T)\bigg)\Bigg] \\ &= E\Bigg[\bigg(\max_{1\leq i\leq n} E\big[\varepsilon_i^2 \boldsymbol{I}\{\|\boldsymbol{a}_{n,i}\||\varepsilon_i|> \epsilon\}|\mathcal{X}\big]\bigg)\bigg(trace(\boldsymbol{M}_k) + o_p\big(n^{\frac{-\mu+1}{\mu+2\nu}}\big)\bigg)\Bigg]. \end{aligned} \tag{6}$$

Let e, $f > 0$, g and h be arbitrary real numbers. It can be proved that

$$E\big[g^2 \boldsymbol{I}\big\{e|g|> h\big\}\big] \leq h^{-f} E\big[|g|^{f+2} e^f\big], \tag{7}$$

(see page 21 of [10]). Using (7), we have

$$E\Bigg[\max_{1\leq i\leq n} E\big[\varepsilon_i^2 \boldsymbol{I}\{\|\boldsymbol{a}_{n,i}\||\varepsilon_i|> \epsilon\}|\mathcal{X}\big]\Bigg] \leq \epsilon^{-2} E[\varepsilon_i^4] E\Bigg[\max_{1\leq i\leq n}\{\|\frac{1}{\sqrt{n}}\boldsymbol{X}_i^{[k]}\|^2\}\Bigg]. \tag{8}$$

Assumption 1 implies that $E[\varepsilon_i^4] < \infty$. From (8), it can be concluded that, if $n \to \infty$, then

$$E\Bigg[\max_{1\leq i\leq n} E\big[\varepsilon_i^2 \boldsymbol{I}\{\|\boldsymbol{a}_{n,i}\||\varepsilon_i|> \epsilon\}|\mathcal{X}\big]\Bigg] \to 0. \tag{9}$$

Assumption 2(a), (6),(9) and Slutsky's theorem, together imply that

$$\sum_{i=1}^{n} E\big[\|\boldsymbol{a}_{n,i}\varepsilon_i\|^2 \boldsymbol{I}\{\|\boldsymbol{a}_{n,i}\varepsilon_i\| > \epsilon\}\big] \to 0, \text{ as } n \to 0. \tag{10}$$

Claim (2) can be promptly derived using (3), (4), (10), Assumption 2(d), Slutsky's theorem and the Lindeberg-Feller central limit theorem (see page 20 of [10]). In order to substitute the conditional proof with an unconditional one, observe that for every $z \in \mathbb{R}$,

$$P(Z_n \leq z) = P(Z_n \leq z|\hat{S} = S)P(\hat{S} = S) + P(Z_n \leq z|\hat{S} \neq S)P(\hat{S} \neq S). \tag{11}$$

Using (11) and (4.2) of [5], for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} P(Z_n \leq z) = \lim_{n\to\infty} P(Z_n \leq z|\hat{S} = S),$$

which completes the proof of Theorem 3.1. □

4. Simulation study

This simulation study aims to evaluate the asymptotic distribution's behavior of

$$Z_n = (\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T)^{-1} \times \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}\varepsilon_i.$$

Independent realizations of the Ornstein–Uhlenbeck processes, with parameters $\theta_u = 5$ and $\sigma_u = 3.5$, were represented by the data $X_1, \dots, X_n$. The response variable Y was derived using the following FLRPI model

$$Y_i = \int_0^1 \beta(t)\, X_i(t)\, dt + \beta_1 X_i(\tau_1) + \beta_2 X_i(\tau_2) + \varepsilon_i, \quad i = 1, \dots, n, \tag{12}$$

where $\beta(t) = 3.5t^3 - 5.5t^2 + 3t + 0.5$, $\tau_1 = 0.25$, $\tau_2 = 0.75$, $\beta_1 = 2$, $\beta_2 = 1$ and $\varepsilon \sim N(0, 1)$. The estimation of impact points relies on the method proposed by [5]. Utilizing the R package bestglm, we determined the optimal number of impact points ($\tilde{S}$) and principal components (k) by conducting best subset selection based on the BIC-criterion for the following model

$$Y_i = \sum_{j=1}^{k} \alpha_j \tilde{Z}_{ij} + \sum_{r=1}^{\tilde{S}} \beta_r X_i(\hat{\tau}_r) + \varepsilon_i^*, \quad i = 1, \ldots, n,$$

where $\beta_r, r = 1, \ldots, \tilde{S}$ and $\alpha_j, j = 1, \ldots, k$ were estimated using the least squares approach.

The density estimates illustrated in Figures 1 and 2 were derived from 1000 simulated data sets. In each figure, the left and right graphs display estimated density of $\big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T\big)^{-1} \times \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}\varepsilon_i$, and the density of $N(\mathbf{0}, \boldsymbol{M}_k^{-1})$, respectively, when the Y_i were generated from Model (12). We considered sample sizes $n = 50$ and 100 in Figures (1) and (2), respectively. As the graphs in Figures (1) and (2) show, there is a tendency for the estimated distribution of

$$Z_n = \big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T\big)^{-1} \times \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}\varepsilon_i$$

to the $N(\mathbf{0}, \boldsymbol{M}_k^{-1})$ as n increases. To investigate the accuracy of our analysis, we utilized the R package MVN and

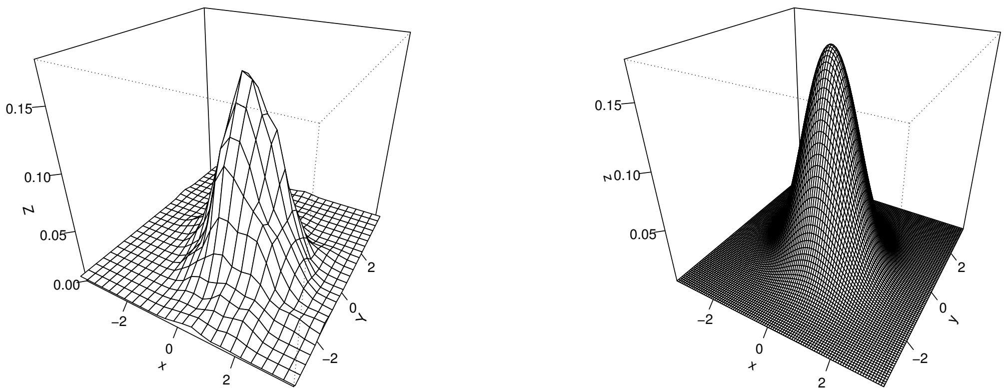

Figure 1: The left and right graphs display the estimated density of $\big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T\big)^{-1} \times \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}\varepsilon_i$ and the density of $N(\mathbf{0}, \boldsymbol{M}_k^{-1})$, respectively, when $n = 50$ and the Y_i were generated from Model (12).

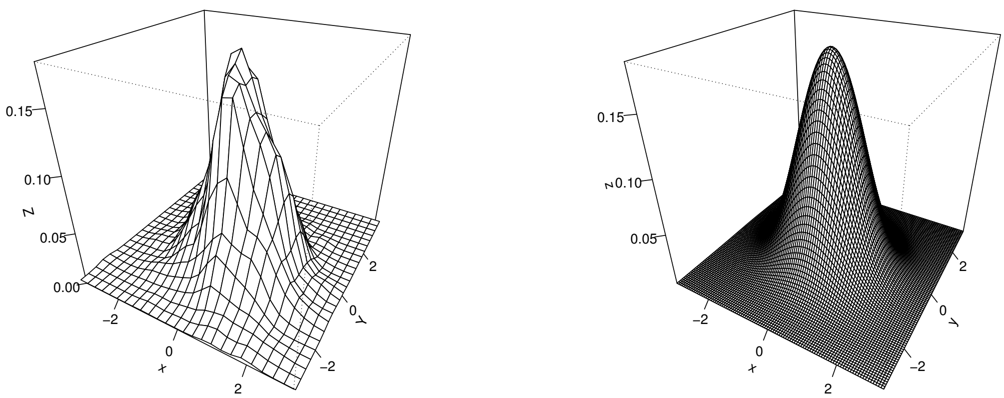

Figure 2: The left and right graphs display the estimated density of $\big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T\big)^{-1} \times \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}\varepsilon_i$ and the density of $N(\mathbf{0}, \boldsymbol{M}_k^{-1})$, respectively, when $n = 100$ and the Y_i were generated from Model (12).

performed five multivariate normality tests on the estimated density of $\big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T\big)^{-1} \times \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{X}_i^{[k]}\varepsilon_i$, when $n = 50$ and 100, and the Y_is were generated from Model (12). The null hypotheses of these tests are the normality of the multivariate distribution. We also calculated univariate descriptive statistics. The results are demonstrated in Tables 1 and 2. As the tables show, for both $n = 50$ and $n = 100$ cases, there is no reason to reject the null hypotheses at the significance level of 0.05. In addition, as the sample size increases, both the mean and skewness of the variables tend to zero, while the kurtosis increases and approaches 3. These observations are consistent with the obtained asymptotic distribution (2).

Table 1: Normality tests results and descriptive statistics for the estimated density of $\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T\right)^{-1}\times\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{X}_i^{[k]}\varepsilon_i$, when $n=50$ and the Y_i were generated from Model (12).

Multivariate Normality	
Test	p-value
Henze-Zirkler	0.282
Doornik-Hansen	0.075
Royston	0.074
Mardia Skewness	0.590
Mardia Kurtosis	0.083

Descriptive Statistics				
	Mean	Std. Dev	Skewness	Kurtosis
Variable 1	0.099	0.963	0.053	2.748
Variable 2	0.139	1.022	0.054	2.826

Table 2: Normality tests results and descriptive statistics for the estimated density of $\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T\right)^{-1}\times\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{X}_i^{[k]}\varepsilon_i$, when $n=100$ and the Y_i were generated from Model (12).

Multivariate Normality	
Test	p-value
Henze-Zirkler	0.115
Doornik-Hansen	0.208
Royston	0.140
Mardia Skewness	0.195
Mardia Kurtosis	0.775

Descriptive Statistics				
	Mean	Std. Dev	Skewness	Kurtosis
Variable 1	0.062	0.948	-0.035	3.022
Variable 2	0.081	0.927	0.041	3.033

5. Conclusion

In this paper, we obtained the asymptotic distribution of $\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{X}_i^{[k]}(\boldsymbol{X}_i^{[k]})^T\right)^{-1}\times\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{X}_i^{[k]}\varepsilon_i$ in Model (1). This is a fundamental result for finding the asymptotic distribution of the impact points coefficients estimators, and then for finding the asymptotic distribution of prediction and providing a statistic for testing the significance of impact points. Our goal in future studies is to investigate these cases.

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