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Original Article

A distributionally robust approach for the risk-parity portfolio selection problem

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ABSTRACT: Risk-parity is one of the most recent and interesting strategies in the portfolio selection area. Considering the mean-standard-deviation risk measure, this paper studies the risk-parity problem under the uncertainty of the covariance matrix. Assuming that the uncertainty is represented by a finite set of scenarios, the problem is formulated as a scenario-based stochastic programming model. Then, since the occurrence probabilities of scenarios are not known with certainty, two ambiguity sets of distributions are considered, and corresponding to each one, a distributionally robust optimization model is presented. Computational experiments on real-world instances taken from the literature confirm the importance of the proposed models in terms of stability, volatility and Sharpe-ratio.

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1. Introduction

Portfolio selection is one the most important research fields in finance and operations research. As one of the pioneer researchers in this field, Markowitz [21] presented a bi-objective optimization model making a trade-off between the maximization of the expected portfolio return and the minimization of its variance. Markowitz model has been a basis for various theories in the portfolio management and many improvements have been proposed in the literature to refine its drawbacks. One of the main drawbacks of the Markowitz model is its sensitivity to input parameters [3], and in this regard, Chopra and Ziemba [10] showed that small changes in the expected return of assets may make massive changes in the optimal values of assets weights obtained by Markowitz model. There are different types of strategies to tackle the estimation error of input parameters. A recently addressed one is the risk budgeting approach the main benefit of which is the stability of asset weights [6]. In the risk budgeting approach, a risk budget is associated with each asset, and the weights of assets in the portfolio are adjusted so that the contribution of each asset to the portfolio risk equals the corresponding risk budget. In particular, if the risk budgets of all assets are equal, the risk-parity portfolio is obtained. Concerning the value of risk budgets, it is worth mentioning that, the risk budget is considered as a hyper-parameter chosen by the investor. Therefore, in most

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studies, risk budgets are assumed to be equal and the risk parity approach is considered. However, Bayat, et al. [22] recently suggested a mechanism to calibrate this hyper-parameter within the optimization model. They provided a bi-level programming problem in which the upper level allocates the risk budgets to assets while supposing some constraints based on the investor's risk priorities, and the lower level determines the risk budgeting portfolio. They showed that the bi-level programming model has a better performance in comparison with the risk parity model in terms of various metrics.

The logarithm-based model introduced by Maillard, et al. [20] is a popular convex formulation for the risk-budgeting approach. This model has been investigated with different risk measures such as standard deviation [4], mean-standard-deviation [24], value at risk [14], and conditional value at risk [5]. Moreover, there exist various extensions for this problem to involve real-world constraints such as putting a lower-bound on the expected portfolio return [16], short selling [11], involving the cardinality constraint [1, 15, 18], and imposing lower and upper bounds on the weight of assets [23].

Although the risk budgeting model is stable in the asset allocation, as stated by Costa and Kwon [12] and Kapsos, et al. [17], the parameters of this problem such as the components of the covariance matrix are unknown and need to be estimated. Hence, the parameters are affected by uncertainty and ignoring this fact may result in deficient performance of the portfolio over out-of-sample data.

To the best of our knowledge, the risk budgeting problem under uncertainty has been just investigated by Costa and Kwon [12], Kapsos, et al. [17] and Costa and Kwon [13]. Costa and Kwon [12] addressed the risk-parity problem with variance risk measure and proposed a robust optimization model by considering the box uncertainty set for the components of the covariance matrix. Later, Costa and Kwon [13] provided a new approach for risk parity problem considering the uncertainty in the probability distributions of assets return and presented a distributionally robust optimization model. Kapsos, et al. [17] proposed the risk budgeting problem with variance risk measure considering a finite set of scenarios for the covariance matrix and presented a scenario-based stochastic programming model with worst-case risk measure so that the contribution of each asset to the variance risk measure under the worst-case scenario equals the given risk budget. Our investigation indicates that the model of Costa and Kwon [12] is just a simple robust formulation, and the model of Kapsos, et al. [17] is conservative due to focusing on the worst scenario. On the other hand, the model of Costa and Kwon [13] does not utilize scenarios. Therefore, in this paper we try to extend the scenario based model of Kapsos, et al. [17] while utilizing the advantages of the distributioally robust optimization model addressed by Costa and Kwon [13].

The main contributions of this paper are as follows: The stochastic risk-parity model of Kapsos, et al. [17] is extended by considering the mean-standard-deviation as a risk measure assuming that a finite set of scenarios is possible for the covariance matrix. Then, since the occurrence probability of scenarios is not known with certainty, the distributionally robust optimization approach which has recently received great attention is utilized [19]. Two well-known ambiguity sets are examined to develop the distributionally robust counterparts. Computational results demonstrate superior performance of our models in comparison with the model of Kapsos, et al. [17] over various datasets.

The remainder of this paper is organized as follows: Section 2 explains some fundamental concepts and defines the risk-parity problem in more detail. In Section 3, first the stochastic risk-parity model of Kapsos, et al. [17] is reviewed, then, our novel scenario-based distributionally robust risk-parity model is presented. In section 4, for two famous ambiguity sets, the robust counterparts are formulated. In section 5 the performance of the robust models is evaluated in comparison with the model Kapsos, et al. [17] over real-world datasets. Section 6 concludes and presents future potential research directions.

2. Risk-parity problem

Let the set $\mathbb{I} = \{1, \ldots, n\}$, indexed by i, include n risky assets. Suppose that decision variable x_i shows the portion of capital invested in asset i where $0 \leq x_i \leq 1$. The vector representation of the assets weights is denoted by $x = (x_1, \ldots, x_n)^T$ and we have $\sum_{i\in\mathbb{I}} x_i = 1$. Let μ and Γ be the mean vector and the covariance matrix associated with assets return, respectively. Consider the mean-standard-deviation as a risk measure which is defined for a given portfolio x as follows:

$$R(x) = -\mu^T x + \alpha\sqrt{(x^T \Gamma x)}. \tag{1}$$

The mean-standard-deviation (1) is one of the most common risk measures, and some other famous risk measures such as value-at-risk and conditional-value-at-risk can be represented in the form of (1) in some special cases [2]. $R(x)$ is positively homogeneous of degree one and continuously differentiable. Additionally, considering SR^+ as an upper bound on the maximum Sharpe ratio, then as $R(x)$ is nonnegative provided that $\alpha > SR^+$ [7]. Therefore, the Euler decomposition theorem implies that:

$$R(x) = x^T \nabla R(x) = \sum_{i \in \mathbb{I}} x_i \frac{\partial R(x)}{(\partial x_i)}, \tag{2}$$

where the term $x_i \frac{\partial R(x)}{(\partial x_i)}$ is defined the total risk contribution of asset i and denoted by RC_i, we have

$$R(x) = \sum_{i \in \mathbb{I}} RC_i.$$

The portfolio x is called the risk-parity portfolio if all assets equally contribute to the portfolio risk. In other words, the risk-parity portfolio satisfies the following equation:

$$\begin{cases} RC_i = \frac{R(x)}{n}, \quad \forall i \in \mathbb{I} \\ \sum_{i \in \mathbb{I}} x_i = 1 \\ x_i \geq 0, \quad \forall i \in \mathbb{I} \end{cases} \tag{3}$$

By means of Karush-Kuhn-Tucker (KKT) conditions, Roncalli [24] proved that in order to find a portfolio x satisfying equation (3), it is enough to solve Model 1 where κ is an arbitrary constant. Assuming that $(y_1^*, \dots, y_n^*)$

$$\begin{aligned} \min \quad & R(y) \\ \text{s.t.} \quad & \sum_{i \in \mathbb{I}} \frac{1}{n} \ln(y_i) \geq \kappa && (4) \\ & y_i \geq 0, \quad \forall i \in \mathbb{I} && (5) \end{aligned}$$

Model 1: Risk-parity problem

is the optimal solution to Model 1 the risk-parity portfolio can be achieved by $x_i^* = \frac{y_i^*}{\sum_{i \in \mathbb{I}} y_i^*}$, $\forall i \in \mathbb{I}$ [24].

Remark 2.1. *As pointed out by [8], the mean-standard-deviation risk measures* (1) *is a convex function. Moreover, due to the convexity of the function $ln(y_i)$, the feasible region of Model 1 is also a convex set. Hence, Model 1 is a convex optimization model and can be solved directly by global nonlinear programming solvers. In our computational experiments, the solver BARON is utilized.*

3. Risk-parity problem with uncertain covariance matrix

In this section, we extend the risk-parity problem under the assumption that the covariance matrix is uncertain and a finite set of scenarios $\mathbb{S} = \{1, \dots, S\}$, indexed by s is possible. The covariance matrix under scenario s and the occurrence probability of s are denoted by Γ_s and P_s respectively. Thus, the mean-standard-deviation risk measure under scenario s, denoted by $R_s(x)$, is defined as follows:

$$R_s(x) = -\mu^T x + \alpha \sqrt{(x^T \Gamma_s x)}. \tag{6}$$

For the Euler decomposition (2) to hold under scenario $s \in S$, the function $R_s(x)$ should be nonnegative. To satisfy this condition, we assume that $\alpha > SR_s^+$ for every $s \in \mathbb{S}$ where SR_s^+ is an upper bound on the maximum Sharpe ratio under scenario s [24].

In what follows, we consider the stochastic risk-parity problem in two cases. The first case has been introduced by Kapsos, et al. [17], and the second case is the innovation of this paper.

3.1. Model of Kapsos, et al. [17]

In the model of Kapsos, et al. [17], the focus is on the worst-case scenario and the aim is to construct a portfolio in which all assets equally contribute to the portfolio risk under the worst-case scenario. It is formulated as follows: Let y^* be the optimal solution to Model 2. Then in the portfolio constructed by $x_i^* = \frac{y_i^*}{\sum_{i \in \mathbb{I}} y_i^*}$ all assets equally contribute to the risk measure R_s under the worst-case scenario.

An equivalent reformulation of Model 2 is as follows and we refer to it as worst-case risk-parity (WRP) problem:

$$\begin{aligned} \min \quad & z = \max_{s \in \mathbb{S}} R_s(y) \\ \text{s.t.} \quad & (4), (5) \end{aligned}$$

Model 2: Stochastic risk-parity model of Kapsos, et al. [17]

$$\begin{aligned} \min \quad & z = \theta \\ \text{s.t.} \quad & (4), (5) \\ & R_s(y) \leq \theta \quad \forall s \in \mathbb{S} \\ & \theta \geq 0 \end{aligned}$$

Model 3: WRP

3.2. Our model

Instead of focusing on the worst-case scenario, in our new model, the aim is to construct a portfolio in which all assets equally contribute to the expectation of the portfolio risk over all scenarios. It is formulated as follows:

$$\begin{aligned} \min \quad & z = \sum_{s \in \mathbb{S}} P_s R_s(y) \\ \text{s.t.} \quad & (4), (5) \end{aligned}$$

Model 4: Our stochastic risk-parity model

Let y^* be the optimal solution to Model 4. Then, in the portfolio constructed by $x_i^* = \dfrac{y_i^*}{\sum_{i \in \mathbb{I}} y_i^*}$ all assets equally contribute to the expected risk measure.

Since the occurrence probabilities of scenarios (i.e. P_s for every $s \in \mathbb{S}$) are not known with certainty, we assume that the probability distribution P belongs to an ambiguity set $\mathbb{P}$ of distributions. Then, the distributionally robust extension of Model 4 is obtained as follows:

$$\begin{aligned} \min \quad & z = \max_{P \in \mathbb{P}} \sum_{s \in \mathbb{S}} P_s \tau_s \\ \text{s.t.} \quad & (4), (5) \end{aligned}$$

Model 5: Extension of Model 4 with distributionally robust approach

Model 5 can be equivalently reformulated as the following model which we refer to as distributionally robust risk parity problem (DRRP): In the next section, we examine two different types of the ambiguity set $\mathbb{P}$ and provide

$$\begin{aligned} \min \quad & z = \max_{P \in \mathbb{P}} \sum_{s \in \mathbb{S}} P_s \tau_s \\ \text{s.t.} \quad & (4), (5) \\ & \tau_s \geq R_s(y) \quad \forall s \in \mathbb{S} && (7) \\ & \tau_s \geq 0 \quad \forall s \in \mathbb{S} && (8) \end{aligned}$$

Model 6: DRRP

the corresponding robust counterparts.

4. Robust counterparts of DRRP for two types of ambiguity set $\mathbb{P}$

In the literature of distributionally robust optimization, the distance-based ambiguity sets are well-known and commonly used. Assuming that a nominal probability distribution is estimated based on the historical data, the distance-based ambiguity sets suppose that the probability distribution belongs to a neighborhood of the nominal probability distribution with a certain distance. Herein, we consider two distance-based ambiguity sets proposed by Zhu and Fukushima [25]: the box ambiguity set and the TV-distance set which are defined in equations (9) and (10), respectively:

$$\mathbb{P}_B = \left\{(P_s = \bar{P}_s + \eta_s \quad \forall s \in \mathbb{S}) : \sum_{s\in\mathbb{S}} \eta_s = 0,\ \eta_s \in [\underline{\eta}_s, \bar{\eta}_s]\ \ \forall s \in \mathbb{S}\right\}, \tag{9}$$

$$\mathbb{P}_{TV} = \left\{(P_s \quad \forall s \in \mathbb{S}) : \frac{1}{2}\sum_{s\in\mathbb{S}} |P_s - \bar{P}_s| \leq \epsilon,\ \sum_{s\in\mathbb{S}} P_s = 1,\ P_s \geq 0\ \ \forall s \in \mathbb{S}\right\}, \tag{10}$$

Where $\bar{P}_s$ is the nominal probability distribution. In box ambiguity set (9), the nominal probability distribution is changed by η_s and a new probability distribution is produced. These changes should be in range $[\underline{\eta}_s, \bar{\eta}_s]$ and constraint $\sum_{s\in\mathbb{S}} \eta_s = 0$ ensures that $\sum_{s\in\mathbb{S}} P_s$ is equal to 1 to meet conditions for probability distribution. Note that the interval $[\underline{\eta}_s, \bar{\eta}_s]$ for all $s \in \mathbb{S}$ is chosen in the way that $\bar{P}_s + \underline{\eta}_s \geq 0$ to have $P_s \geq 0$. Moreover, in TV-distance set (10), contains probability distribution P_s whose l_1-distance to the nominal probability distribution $\bar{P}_s$ is less than certain parameter 2ϵ. Other constraints in set $\mathbb{P}_{TV}$ guarantee the conditions for probability distribution. The robust counterparts of the model DRRP associated with ambiguity sets (9) and (10) are provided in subsections 4.1 and 4.2, respectively.

4.1. DRRP with box ambiguity set

Considering the box ambiguity set (9), the term $\sum_{s\in\mathbb{S}} P_s\tau_s$ in the objective function of the model DRRP can be restated as follows:

$$\sum_{s\in\mathbb{S}} P_s\tau_s = \sum_{s\in\mathbb{S}} (\bar{P}_s + \eta_s)\tau_s = \sum_{s\in\mathbb{S}} \bar{P}_s\tau_s + \sum_{s\in\mathbb{S}} \eta_s\tau_s. \tag{11}$$

By substituting equation (11) in the objective function of the model DRRP, we get the following model: In Model 7,

$$\begin{aligned} \min \quad & z = \sum_{s\in\mathbb{S}} \bar{P}_s\tau_s + \max_{\eta_s\ \forall s\in\mathbb{S}} \left\{ \sum_{s\in\mathbb{S}} \eta_s\tau_s \ :\ \sum_{s\in\mathbb{S}} \eta_s = 0,\ \eta_s \in [\underline{\eta}_s \bar{\eta}_s]\ \ \forall s \in \mathbb{S} \right\} \\ \text{s.t.} \quad & (4), (5), (7), (8) \end{aligned}$$

Model 7: Restatement of the model DRRP for the box ambiguity set

the inner maximization problem can be substituted by its dual form. Then, we get the following model which we refer to as box based counterpart of DRRP (B-DRRP):

$$\begin{aligned} \min \quad & z = \sum_{s\in\mathbb{S}} \bar{P}_s\tau_s + \bar{\eta}_s v_s - \underline{\eta}_s v'_s \\ \text{s.t.} \quad & (4), (5), (7), (8) \\ & w + v_s - v'_s = \tau_s \quad \forall s \in \mathbb{S} && (12) \\ & w \in \mathbb{R} && (13) \\ & v_s, v'_s \geq 0 \quad \forall s \in \mathbb{S} && (14) \end{aligned}$$

Model 8: B-DRRP

4.2. DRRP with TV-distance set

By substituting TV-distance set (10) in Model DRRP, we get the following model: The inner maximization problem in Model 9 can be equivalently restated as the following model in which the absolute value functions are linearized by constraints (16)-(19): By substituting the dual form of Model 10 into Model 9, we get the TV-distance based counterpart of DRRP (TV-DRRP):

$$\begin{aligned} \min \quad & z = \max_{P_s, \forall s \in \mathbb{S}} \Big\{ \sum_{s \in \mathbb{S}} P_s \tau_s : \frac{1}{2} \sum_{s \in \mathbb{S}} |P_s - \bar{P}_s| \leq \epsilon, \ \sum_{s \in \mathbb{S}} P_s = 1, \ P_s \geq 0 \ \ \forall s \in \mathbb{S} \Big\} \\ \text{s.t.} \quad & (4), (5), (7), (8) \end{aligned}$$

Model 9: Restatement of the model DRRP for the TV-distance set

$$\begin{aligned} \min \quad & \sum_{s \in \mathbb{S}} P_s \tau_s \\ \text{s.t.} \quad & \sum_{s \in \mathbb{S}} P_s = 1 && (15) \\ & \sum_{s \in \mathbb{S}} a_s \leq 2\epsilon && (16) \\ & P_s - a_s \leq \bar{P}_s \quad \forall s \in \mathbb{S} && (17) \\ & -P_s - a_s \leq -\bar{P}_s \quad \forall s \in \mathbb{S} && (18) \\ & P_s \geq 0, a_s \geq 0 \quad \forall s \in \mathbb{S} && (19) \end{aligned}$$

Model 10: Linear reformulation of the inner maximization problem

$$\begin{aligned} \min \quad & q + 2\epsilon b + \sum_{s \in \mathbb{S}} (\bar{P}_s u_s + \bar{P}_s u'_s) \\ \text{s.t.} \quad & (4), (5), (7), (8) \\ & q_s + u_s - u'_s \geq \tau_s \quad \forall s \in \mathbb{S} && (20) \\ & b_s - u_s + u'_s \geq 0 \quad \forall s \in \mathbb{S} && (21) \\ & q \in \mathbb{R}, b \geq 0 && (22) \\ & u_s, u'_s \geq 0 \quad \forall s \in \mathbb{S} && (23) \end{aligned}$$

Model 11: TV-DRRP

It is worth mentioning that model WRP finds a portfolio only based on the worst-case risk measure which is actually according to the worst scenario and ignores other scenarios. However, these scenarios may have some useful information. On the other hand, model DRRP with various ambiguity sets uses the advantage of all scenarios and gives them occurrence probabilities. In the computational experiments section, to evaluate models B-DRRP and TV-DRRP, we compare them with the model WRP. We indicate that models B-DRRP and TV-DRRP have a better performance in different aspects such as robustness of the portfolio, standard deviation and Sharpe ratio on out-of-sample data in comparison with model WRP.

5. Computational experiments

In this section, the importance of models B-DRRP and TV-DRRP are investigated in comparison with model WRP on real-world instances from the market indices FTSE100 and S&P500 taken from Bruni, et al. [9]. All computational experiments are performed on a laptop running Windows 10, Core i7 processor and 8 GB RAM. In additional, all models are coded in GAMS [7] and to solve them, we have used BARON solver.

Scenarios are generated based on historical data of asset returns and included in-sample and out-of-sample scenarios. Based on historical data, we totally generate $S+1$ scenarios. According to [17], Historical data is divided into $S+1$ consecutive periods which are contained assets return information in the β consecutive time. The covariance matrix corresponding to each period s is estimated and denoted by Γ_s. Scenarios $1, \ldots, S$ are considered as in-sample scenarios with occurrence probability $P_s = \frac{1}{S}$ and models B-DRRP, TV-DRRP and WRP are solved with these scenarios. Scenario $S+1$ is an out-of-sample scenario which is in fact the future of in sample scenarios and it is utilized for evaluating models. See Figure 1.

Note that as pointed out by Zhu and Fukushima [25],to avoid highly conservative decisions, the distance between the probability distributions and the nominal distribution should not exceed a given threshold. Therefore, our

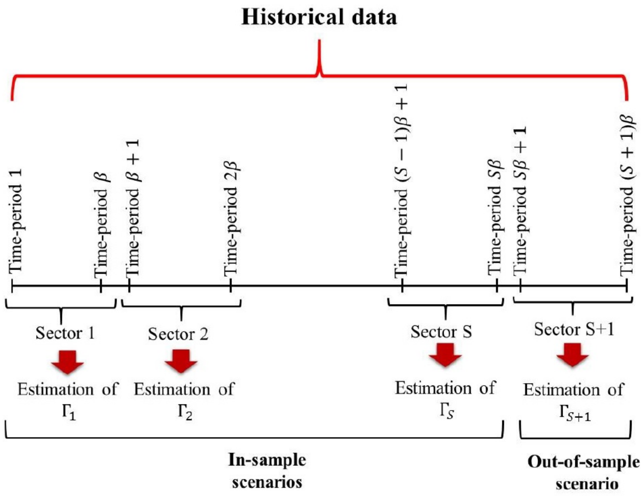

Figure 1: Generating scenarios based on historical data

preliminary experiments indicate that the setting $\underline{\eta}_s = \dfrac{-0.2}{S}$ and $\bar{\eta}_s = \dfrac{0.2}{S}$ for set $\mathbb{P}_B$ and $\epsilon = 0.15$ for set $\mathbb{P}_{TV}$ are appropriate choices.

Let $x^m = (x_1^m, \ldots, x_n^m)$ be the optimal portfolio obtained by the model $m \in \{\text{WRP,B-DRRP,TV-DRRP}\}$ Additionally, consider $x^{out} = (x_1^{out}, \ldots, x_n^{out})$ indicate the optimal risk-parity portfolio obtained by solving Model 1 for out-of-sample scenario Γ_{s+1}. The index D_m shows absolute deviation between portfolio x^m and x^{out}, and indicates how much the portfolio x^m is close to the ideal out-of sample portfolio x^{out}. It is clear that the smaller values of this index are more desirable and show the stability of the obtained portfolio in the out-of-sample scenario.

$$D_m = \frac{\sum_{i \in \mathbb{I}} |x_i^{\text{out}} - x_i^m|}{n}.$$

Other indices which can be used to evaluate the portfolio obtained by each model $m \in \{\text{WRP,B-DRRP,TV-DRRP}\}$ are the volatility and the Sharpe ratio under the out-of-sample scenario, denoted by VOL_m and SR_m, respectively.

$$VOL_m = \sqrt{(x^m)^T \Gamma_{S+1} x^m}, \quad SR_m = \frac{\mu^T x^m}{VOL_m}.$$

Table 1 compares the performance of models B-DRRP, TV-DRRP and WRP based on the indices D_m, VOL_m and SR_m on various instances. The columns labeled by $|\mathbb{I}|$, $|\mathbb{S}|$ and "Market", respectively, represent the number of assets and scenarios as well as the market-index from which the information is extracted. In each row, the best results are shown in bold.

Table 1: Investigating the performance of models B-DRRP, TV-DRRP and WRP over out-of-sample scenario

Row	Characteristics		Market	VOL_m			SR_m			D_m		
	$\lvert\mathbb{S}\rvert$	$\lvert\mathbb{I}\rvert$		VOL_{B-DRRP}	$VOL_{TV-DRRP}$	VOL_{WRP}	SR_{B-DRRP}	$SR_{TV-DRRP}$	SR_{WRP}	D_{B-DRRP}	$D_{TV-DRRP}$	D_{WRP}
1	5	8	FTSE100	**0.0134**	**0.0134**	0.0135	0.2104	0.2138	**0.2220**	**0.0460**	0.0485	0.0571
2	5	8	FTSE100	**0.0142**	**0.0142**	**0.0142**	**0.2524**	0.2341	0.2505	**0.0704**	0.0708	0.0762
3	5	8	S&P500	0.0294	0.0289	**0.0280**	-0.0513	-0.0500	**-0.0469**	0.0371	**0.0323**	0.0496
4	5	8	S&P500	0.0215	0.0215	**0.0210**	-0.2480	-0.2527	-0.2295	**0.0340**	0.0380	0.0367
5	10	12	FTSE100	0.0115	0.0115	**0.0113**	**0.2870**	0.2474	0.2601	**0.0208**	0.0222	0.0311
6	10	12	FTSE100	**0.0267**	0.0268	0.0273	**0.0023**	-0.0004	-0.0075	**0.0220**	0.0231	0.0309
7	10	12	S&P500	**0.0450**	0.0451	0.0456	**0.1171**	0.1170	0.1141	**0.0196**	0.0203	0.0257
8	10	12	S&P500	0.0563	0.0529	**0.0504**	**0.1061**	0.1002	0.0970	0.0284	0.0241	**0.0198**
9	15	15	FTSE100	0.0322	**0.0242**	0.0432	0.0062	**0.2738**	-0.0162	**0.0184**	0.0544	0.0272
10	15	15	FTSE100	**0.0238**	0.0339	0.0245	0.2712	0.0001	**0.2815**	0.0542	**0.0204**	0.0547
11	15	15	S&P500	0.0191	**0.019**	0.0193	**0.2642**	0.2560	0.2129	**0.0132**	0.0138	0.0229
12	15	15	S&P500	**0.0200**	0.0201	**0.0200**	0.2680	0.2767	**0.2971**	**0.0107**	0.0123	0.0199
13	20	30	FTSE100	0.0136	**0.0135**	0.0145	0.1491	**0.1614**	0.1040	**0.0180**	0.0189	0.0273
14	20	30	FTSE100	0.0136	0.0138	**0.0135**	**0.1696**	0.1359	0.1371	**0.0190**	0.0197	0.0217
15	20	30	S&P500	**0.0172**	0.0174	0.0174	0.3069	0.3064	**0.3258**	0.0152	**0.0146**	0.0193
16	20	30	S&P500	**0.0170**	0.0171	0.0173	0.3123	**0.3129**	0.3108	**0.0194**	**0.0194**	0.0215
17	30	50	FTSE100	**0.0114**	0.0151	0.0116	**0.2706**	0.1225	0.2658	0.0301	**0.0130**	0.0335
18	30	50	FTSE100	0.0149	**0.0117**	0.0150	0.1133	**0.2731**	0.1304	**0.0118**	0.0307	0.0163
19	30	50	S&P500	0.0102	**0.0100**	0.0101	-0.0475	-0.0395	**-0.0125**	**0.0318**	0.0319	0.0342
20	30	50	S&P500	**0.0150**	**0.0150**	0.0151	0.0892	0.0888	**0.0999**	**0.0182**	0.0184	0.0200
Ave.				0.021300	**0.021255**	0.021640	**0.142455**	0.138875	0.139820	**0.026915**	0.027340	0.032280

As can be seen in Table 1, in comparison to the model WRP of Kapsos, et al. [17], our proposed distributionally robust models B-DRRP and TV-DRRP have better performance in almost all instances regarding the index D_m. This confirms that the portfolios obtained by B-DRRP and TV-DRRP are more stable and closer to the ideal out-of-sample portfolio x^{out}. Further, the averaged results reported in the last row of Table 1 indicate that the models B-DRRP and TV-DRRP performs better than the model WRP regarding all indices, on average.

6. Conclusions

In this paper, the risk-parity portfolio selection problem under the assumption of uncertainty in the covariance matrix was formulated as a scenario based distributionally robust optimization model. Then, for two given ambiguity sets, the robust counterparts were provided. The proposed models were compared with the risk-parity model of Kapsos, et al. [17] on out-of-sample scenario and their proper performance was shown in terms of standard deviation, Sharpe ratio, and the stability index. The extension of the proposed models to incorporate the real-world constraints (such as cardinality constraints) is suggested for future work.

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