AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 6(1) (2025) 31-39
https://doi.org/10.22060/AJMC.2023.22211.1140

Original Article

# $C^*$-algebra-valued $S_b$-metric spaces and applications to integral equations

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**ABSTRACT:** We first introduce the concept of $C^*$-algebra-valued $S_b$-metric space, then we prove Banach contraction principle in this space. Finally, existence and uniqueness results for one type of integral equation is discussed.

**Review History:**

Received:17 February 2023
Revised:14 November 2023
Accepted:14 November 2023
Available Online:01 December 2024

**Keywords:**

Banach contraction principle
$b$-metric space
$S_b$-metric space
$C^*$-algebra
Integral equation

**MSC (2020):**

34A12; 47H10; 54H25

## 1. Introduction

In 1922, Stefan Banach established a significant fixed point theorem known as the "Banach Contraction Principle (BCP)" which is one of the fundamental results in analysis and serves as an essential axiom of fixed point theory. The BCP has attracted the attention of many mathematicians, leading to various applications and extensions of this principle.

In 1993, a new origin for semimetric spaces was introduced by Czerwik [3]. Since then, numerous authors have studied fixed point theory in such spaces [1, 2, 5, 14]. Additionally, Xia [19] referred to these spaces as $b$-metric space. For more details on this space, see [6].

Recently, in [8], the authors introduced the notion of $C^*$-algebra-valued metric spaces. In fact, the study of the set of real numbers has transitioned to the framework of all positive elements of a unital $C^*$-algebra. In [7], as a generalization of $b$-metric spaces and operator-valued metric spaces [9], the authors introduced a new type of metric spaces, namely, $C^*$-algebra-valued $b$-metric spaces, and provided some fixed point results for self-maps satisfying contractive conditions in such spaces.

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Lately, Sedghi et al. [15] introduced the concept of an $S$-metric space as a generalization of the $G$-metric space [12] and $D^*$-metric space [16]. Following this, many authors have generalized $S$-metric spaces and obtained various results related to the existence of fixed points, see [10, 13, 17]

Inspired by [1], the authors in [18] motivated the study of $S_b$-metric space as a generalization of $b$-metric spaces and presented some fixed point results under various natures of contractions in complete $S_b$-metric spaces.

In [4], the authors introduced $C^*$-algebra-valued $S$-metric spaces and proved Banach contraction along with a coupled fixed point theorem in such spaces.

In this paper, we first introduce $C^*$-algebra-valued $S_b$-metric spaces and present some fixed point results for maps defined in this space. Finally, we choose the problem of existence and uniqueness of solutions of a specific type of integral equation to demonstrate the results detailed in the paper.

## 2. Basic Definitions

For the reader's convenience, we recall the following definitions and notations which will be needed in the sequel:

We start by reviewing some facts about $C^*$-algebras [11]; Suppose that $\mathcal{A}$ is an unital $C^*$-algebra with unit $I$. Set $\mathcal{A}_h = \{a \in \mathcal{A} : \ a = a^*\}$. We say that $a \in \mathcal{A}$ is a positive element, denote by $a \geq 0_{\mathcal{A}}$, if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$, where $0_{\mathcal{A}}$ is the zero element in $\mathcal{A}$ and $\sigma(a)$ is the spectrum of $a$.
There is a natural partial ordering on $\mathcal{A}_h$ given by $a \leq b$ if and only if $b - a \geq 0_{\mathcal{A}}$. From now on, we will denote by $\mathcal{A}_+$ the set $\{a \in \mathcal{A} : \ a \geq 0_{\mathcal{A}}\}$ and by $\mathcal{A}'$ the set $\{a \in \mathcal{A} : ab = ba\,,\ \forall b \in \mathcal{A}\}$.

**Definition 2.1.** *[8] Suppose $\mathcal{X}$ be a nonempty set, and $\mathcal{A}$ be a $C^*$-algebra. Let $A \in \mathcal{A}'$ be such that $A \geq I$. A mapping $d_b : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ is called a $C^*$-algebra-valued b-metric on $\mathcal{X}$ if the following conditions hold for all $x, y, z \in \mathcal{A}$:*

*(1) $d_b(x, y) \geq 0_{\mathcal{A}}$ for all $x$ and $y$ in $\mathcal{X}$ and $d_b(x, y) = 0 \Leftrightarrow x = y$;*

*(2) $d_b(x, y) = d_b(y, x)$;*

*(3) $d_b(x, y) \leq A[d_b(x, z) + d_b(z, y)]$.*

*The triplet $(\mathcal{X}, \mathcal{A}, d_b)$ is called a $C^*$-algebra-valued b-metric space with coefficient $A$.*

**Definition 2.2.** *[4] Let $\mathcal{X}$ be a nonempty set. We say that a mapping $S : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ is $C^*$-algebra-valued $S$-metric if, for every $x, y, z, a \in \mathcal{X}$ we have:*

*(1) $S(x, y, z) \geq 0_{\mathcal{A}}$;*

*(2) $S(x, y, z) = 0_{\mathcal{A}}$ if and only if $x = y = z$;*

*(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.*

*furthermore $(\mathcal{X}, \mathcal{A}, S)$ is called a $C^*$-algebra-valued $S$-metric spaces.*

**Definition 2.3.** *[18] Let $\mathcal{X}$ be a nonempty set and $s \geq 1$ be a given number. A function $S_b : \mathcal{X}^3 \to [0, \infty)$ is said to be a $S_b$-metric if and only if for all $x, y, z, t \in \mathcal{X}$, the following conditions hold:*

*(1) $S_b(x, y, z) = 0$ if and only if $x = y = z$;*

*(2) $S_b(x, y, z) \leq s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]$*

*the pair $(\mathcal{X}, S_b)$ is called an $S_b$-metric space.*

**Definition 2.4.** *A $S_b$-metric $S_b$ is said to be symmetric if*

$$S_b(x, x, y) = S_b(y, y, x), \quad \textit{for all } x, y \in \mathcal{X}.$$

Now, we define the concept of $C^*$-algebra-valued $S_b$-metric space. Its concept is motivated by the above notions:

**Definition 2.5.** *Let $\mathcal{X}$ be a nonempty set and $A \in \mathcal{A}'$ such that $A \geq I$. Let the mapping $S_b : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfies:*

*(1) $S_b(x, y, z) \geq 0_{\mathcal{A}}$ for all $x, y, z \in \mathcal{X}$;*

*(2) $S_b(x, y, z) = 0_{\mathcal{A}}$ if and only if $x = y = z$;*

*(3)* $S_b(x,y,z) \leq A[S_b(x,x,a)+S_b(y,y,a)+S_b(z,z,a)]$ *for all* $x,y,z,a \in \mathcal{X}$.

*Then* $S_b$ *is said to be* $C^*$*-algebra-valued* $S_b$*-metric on* $\mathcal{X}$ *and* $(\mathcal{X},\mathcal{A},S_b)$ *is said to be a* $C^*$*-algebra valued* $S_b$*-metric space.*

Now, we give an important property which played a major role in the study of the new space:

**Definition 2.6.** *A* $C^*$*-algebra-valued* $S_b$*-metric* $S_b$ *is said to be symmetric if*

$$S_b(x,x,y) = S_b(y,y,x), \quad for\ all\ x,y \in \mathcal{X}.$$

**Example 2.1.** *Let* $X = \mathbb{R}$ *and* $\mathcal{A} = M_2(\mathbb{R})$ *be all* $2\times 2$*-matrices with the usual operations of addition, scalar multiplication and matrix multiplication. It is clear that*

$$\|A\| = \Big(\sum_{i,j=1}^{2} |a_{ij}|^2\Big)^{\frac{1}{2}},$$

*defines a norm on* $\mathcal{A}$ *where* $A=(a_{ij}) \in \mathcal{A}$. $*: \mathcal{A} \to \mathcal{A}$ *defines an involution on* $\mathcal{A}$ *where* $\mathcal{A}^* = \mathcal{A}$. *Then* $\mathcal{A}$ *is a* $C^*$*-algebra. For* $A=(a_{ij})$ *and* $B=(b_{ij})$ *in* $\mathcal{A}$*, a partial order on* $\mathcal{A}$ *can be given as follows:*

$$A \leq B \Leftrightarrow (a_{ij}-b_{ij}) \leq 0, \quad \forall i,j = 1,2.$$

*Let* $(X,d)$ *be a b-metric space with* $b \geq 1$ *and* $S_b : X\times X\times X \to M_2(\mathbb{R})$ *be defined by*

$$S_b(x,y,z) = \begin{bmatrix} d(x,z)+d(y,z) & 0 \\ 0 & d(x,z)+d(y,z) \end{bmatrix},$$

*then it is a* $C^*$*-algebra-valued* $S_b$*-metric space. Now, we check the condition (3) of Definition 2.5:*

$$\begin{aligned} S_b(x,y,z) &= \begin{bmatrix} d(x,z)+d(y,z) & 0 \\ 0 & d(x,z)+d(y,z) \end{bmatrix} \\ &\leq b \begin{bmatrix} d(x,a)+d(z,a) & 0 \\ 0 & d(x,a)+d(z,a) \end{bmatrix} + b \begin{bmatrix} d(y,a)+d(z,a) & 0 \\ 0 & d(y,a)+d(z,a) \end{bmatrix} \\ &= b \begin{bmatrix} d(x,a) & 0 \\ 0 & d(x,a) \end{bmatrix} + b \begin{bmatrix} d(y,a) & 0 \\ 0 & d(y,a) \end{bmatrix} + 2b \begin{bmatrix} d(z,a) & 0 \\ 0 & d(z,a) \end{bmatrix} \\ &\leq 2b \begin{bmatrix} d(x,a) & 0 \\ 0 & d(x,a) \end{bmatrix} + 2b \begin{bmatrix} d(y,a) & 0 \\ 0 & d(y,a) \end{bmatrix} + 2b \begin{bmatrix} d(z,a) & 0 \\ 0 & d(z,a) \end{bmatrix} \\ &= b[2 \begin{bmatrix} d(x,a) & 0 \\ 0 & d(x,a) \end{bmatrix} + 2 \begin{bmatrix} d(y,a) & 0 \\ 0 & d(y,a) \end{bmatrix} + 2 \begin{bmatrix} d(z,a) & 0 \\ 0 & d(z,a) \end{bmatrix}] \\ &= b[S_b(x,x,a)+S_b(y,y,a)+S_b(z,z,a)], \end{aligned}$$

*for all* $x,y,z \in X$*. So* $(X,\mathcal{A},S_b)$ *is a* $C^*$*-algebra-valued* $S_b$*-metric space.*

**Example 2.2.** *Let* $X=\mathbb{R}$ *and* $A = M_2(\mathbb{R})$ *and* $(X,d)$ *be a metric space. Let the function* $S_b : X\times X\times X \to A$ *be defined as:*

$$S_b(x,y,z) = \begin{bmatrix} (d(x,y)+d(y,z)+d(x,z))^p & 0 \\ 0 & (d(x,y)+d(y,z)+d(x,z))^p \end{bmatrix},$$

*where* $p > 1$ *and* $x,y,z \in X$ *. For* $A=(a_{ij})$ *and* $B=(b_{ij})$ *in* $A$*, a partial order on* $A$ *can be given as follows:*

$$A \leq B \Leftrightarrow (a_{ij}-b_{ij}) \leq 0, \quad \forall i,j = 1,2.$$

*It can be shown that* $(X,A,S_b)$ *is an* $C^*$*-algebra-valued* $S_b$*-metric with* $b = 2^{3(p-1)}$*, but* $(X,A,S_b)$ *is not necessarily a* $C^*$*-algebra-valued* $S$*-metric.*

**Definition 2.7.** *Suppose that* $(\mathcal{X},\mathcal{A},S_b)$ *is a* $C^*$*-algebra-valued symmetric* $S_b$*-metric space. We call a mapping* $T : \mathcal{X} \to \mathcal{X}$ *a* $C^*$*-algebra-valued contractive mapping on* $\mathcal{X}$*, if there exists a* $B \in \mathcal{A}$ *with* $\|B\| < 1$ *such that*

$$S_b(Tx,Tx,Ty) \leq B^* S_b(x,x,y) B, \quad \forall x,y \in \mathcal{X}. \tag{1}$$

## 3. Main Results

**Theorem 3.1.** *Let $(\mathcal{X}, \mathcal{A}, S_b)$ be a $C^*$-algebra-valued symmetric $S_b$-metric space and let $T : \mathcal{X} \to \mathcal{X}$ is a contractive mapping. Then there exists a unique fixed point in $\mathcal{X}$.*

**Proof.** If $B = 0$, $T$ maps $\mathcal{X}$ into a single point. Assume, without loss of generality, that $B \neq 0$.

Choose an $x_0 \in \mathcal{X}$ and set $x_{n+1} = Tx_n = \cdots = T^{n+1}x$ for $n = 1, 2, \ldots$. For convenience, denote by $B_0$ the element $S_b(x, x, Tx)$ in $\mathcal{A}$. Also, in a $C^*$-algebra, for $a, b \in \mathcal{A}_+$ with $a \leq b$, and for any $x \in \mathcal{A}$, both $x^*ax$ and $x^*bx$ are positive elements and $x^*ax \leq x^*bx$. First we focus on existence of a fixed point. We choose $x \in \mathcal{X}$ and show that $\{T^n(x)\}$ is a Cauchy sequence with respect to $\mathcal{A}$. By using induction, we have:

$$\begin{aligned}
S_b(x_n, x_n, x_{n+1}) &= S_b(T^n x, T^n x, T^{n+1}x) \\
&\leq B^* S_b(T^{n-1}x, T^{n-1}x, T^n x)B \\
&\leq (B^*)^2 S_b(T^{n-2}x, T^{n-2}x, T^{n-1}x)B^2 \\
&\leq \ldots \\
&\leq (B^*)^n S_b(x, x, Tx)B^n \\
&= (B^n)^* B_0 B^n.
\end{aligned}$$

Therefore for any $n \geq 1$ and $p \geq 1$, it follows that

$$\begin{aligned}
S_b(x_n, x_n, x_{n+p}) &= S_b(T^n x, T^n x, T^{n+p}x) \\
&\leq A(S_b(T^n, T^n, T^{n+1}) + S_b(T^n, T^n, T^{n+1}) + S_b(T^{n+p}, T^{n+p}, T^{n+1})) \\
&\leq 2AS_b(T^n, T^n, T^{n+1}) + A(2AS_b(T^{n+p}, T^{n+p}, T^{n+2}) + AS_b(T^{n+1}, T^{n+1}, T^{n+2})) \\
&\leq 2AS_b(T^n, T^n, T^{n+1}) + A^2 S_b(T^{n+1}, T^{n+1}, T^{n+2}) \\
&\qquad + 2A^2(2AS_b(T^{n+p}, T^{n+p}, T^{n+3}) + AS_b(T^{n+2}, T^{n+2}, T^{n+3})) \\
&\vdots \\
&\leq 2A^1(B^*)^n B_0(B)^n + 2^0 A^2(B^*)^{n+1}B_0B^{n+1} + 2A^3(B^*)^{n+2}B_0B^{n+2} + 2^2A^4(B^*)^{n+3}B_0B^{n+3} \\
&\qquad + 2^3A^5(B^*)^{n+4}B_0B^{n+4} + \cdots + 2^{p-2}A^p(B^*)^{n+p-1}B_0(B)^{n+p-1} \\
&= 2A(B^*)^n B_0 B^n + \sum_{k=2}^{p} 2^{k-2}A^k(B^*)^{n+k-1}B_0(B)^{n+k-1} \\
&= 2((B^*)^n A^{\frac{1}{2}} B_0{}^{\frac{1}{2}})(B_0{}^{\frac{1}{2}} A^{\frac{1}{2}} B^n) + \sum_{k=2}^{p} 2^{k-2}((B^*)^{n+k-1} A^{\frac{k}{2}} B_0{}^{\frac{1}{2}})(B_0{}^{\frac{1}{2}} A^{\frac{k}{2}} B^{n+k-1}) \\
&= 2(B_0{}^{\frac{1}{2}} A^{\frac{1}{2}} B^n)^*(B_0{}^{\frac{1}{2}} A^{\frac{1}{2}} B^n) + \sum_{k=2}^{p} 2^{k-2}(B_0{}^{\frac{1}{2}} A^{\frac{k}{2}} B^{n+k-1})^*(B_0{}^{\frac{1}{2}} A^{\frac{k}{2}} B^{n+k-1}) \\
&= 2|B_0{}^{\frac{1}{2}} A^{\frac{1}{2}} B^n|^2 + \sum_{k=2}^{p} 2^{k-2}|B_0{}^{\frac{1}{2}} A^{\frac{k}{2}} B^{n+k-1}|^2 \\
&\leq 2\|B_0{}^{\frac{1}{2}} A^{\frac{1}{2}} B^n\|^2 I + \sum_{k=2}^{p} 2^{k-2}\|B_0{}^{\frac{1}{2}} A^{\frac{k}{2}} B^{n+k-1}\|^2 I \\
&\leq 2\|B_0{}^{\frac{1}{2}}\|^2 \|A^{\frac{1}{2}}\|^2 \|B^n\|^2 I + \|B_0{}^{\frac{1}{2}}\|^2 \sum_{k=2}^{p} 2^{k-2}\|B\|^{2(n+k-1)}\|A\|^k I \\
&= 2\|B_0\|\|A^{\frac{1}{2}}\|^2\|B^n\|^2 I + \|B_0\| \sum_{k=1}^{p-1} 2^{k-1}\|B\|^{2(n+k)}\|A\|^{k+1} I \\
&\leq 2\|B_0\|\|A^{\frac{1}{2}}\|^2\|B\|^{2n} I + \|B_0\| 2^{k-1} \frac{\|B\|^{2(n+p)}\|A\|^p}{\|B\|^2 - \|B\|^4\|A\|} I \longrightarrow 0 \ (n \to \infty).
\end{aligned}$$

Where $B_0 = S_b(x, x, Tx)$. So $\{T^n(x)\}$ is a Cauchy sequence with respect to $\mathcal{A}$, and by completeness of $(\mathcal{X}, \mathcal{A}, S_b)$

there exists $x_0 \in \mathcal{X}$ such that $\lim_{n\to\infty} T^n(x) = x_0$. Since

$$\begin{aligned} 0 &= S_b(Tx_0, Tx_0, x_0) \\ &\le B^* S_b(T^{n-1}x, T^{n-1}x, T^n x)B \\ &\le A[S_b(Tx_0, Tx_0, Tx_n) + S_b(Tx_0, Tx_0, Tx_n) + S_b(x_0, x_0, x_n)] \\ &\le AB^* S_b(x_0, x_0, x_n)B + AB^* S_b(x_0, x_0, x_n)B + AS_b(x_0, x_0, x_n) \longrightarrow 0 \ (n \to \infty), \end{aligned}$$

we thus conclude that $Tx_0 = x_0$, i.e., $x_0$ is a fixed point of $T$.

Now suppose there exist $u, v \in \mathcal{X}$ such that $u = T(u)$ and $v = T(v)$. Since $T$ is a $C^*$-algebra-valued contractive mapping, we have

$$0 \le S_b(u, u, v) = S_b(Tu, Tu, Tv) \le B^* S_b(u, u, v)B,$$

Since $\|B\| < 1$, we have

$$\begin{aligned} 0 &\le \|S_b(u, u, v)\| = \|S_b(Tu, Tu, Tv)\| \\ &\le \|B^* S_b(u, u, v)B\| \\ &\le \|B^*\|\|S_b(u, u, v)\|\|B\| \\ &= \|B\|^2 \|S_b(u, u, v)\| \\ &< \|S_b(u, u, v)\|. \end{aligned}$$

However, this is impossible, so $S_b(u, u, v) = o$ and $u = v$. Thus the fixed point is unique. □

The example below is a simple example about Theorem 3.1:

**Example 3.1.** *Let $(X, \mathcal{A}, S_b)$ be as in Example 2.1. Define a map $T : X \to X$ by $T(x) = \frac{x}{8}$. Since*

$$\begin{aligned} S_b(Tx, Tx, Ty) &= S_b(\frac{x}{8}, \frac{x}{8}, \frac{y}{8}) \\ &= \begin{bmatrix} \frac{1}{4}|x-y| & 0 \\ 0 & \frac{1}{4}|x-y| \end{bmatrix} \\ &\le \begin{bmatrix} \frac{1}{\sqrt{8}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix} \begin{bmatrix} 2|x-y| & 0 \\ 0 & 2|x-y| \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{8}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}, \end{aligned}$$

*where $A = \begin{bmatrix} \frac{1}{\sqrt{8}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}$ and $\|A\| = \frac{1}{2} < 1$, then $T$ is a $C^*$-algebra-valued contractive mapping. Moreover, $T$ satisfies the hypothesis of Theorem 3.1. so $0$ is the unique fixed point of $T$.*

We now have the following extension of Theorem 3.1.

**Corollary 3.2.** *Let $(\mathcal{X}, \mathcal{A}, S_b)$ be a complete $C^*$-algebra valued $S_b$-metric space. Suppose $T : \mathcal{X} \to \mathcal{X}$ satisfies*

$$\|S_b(Tx, Tx, Ty)\| \le \|B\|\|S_b(x, x, y)\|,$$

*where $B \in \mathcal{A}'_+$ with $\|B\| < 1$, for all $x, y \in \mathcal{X}$. Then there exists a unique fixed point in $\mathcal{X}$.*

**Theorem 3.3.** *Let $(\mathcal{X}, \mathcal{A}, S_b)$ be a complete $C^*$-algebra-valued $S_b$-metric space. Suppose the mapping $T : \mathcal{X} \to \mathcal{X}$ satisfies:*

$$S_b(Tx, Ty, Tz) \le B(S_b(x, x, Tx) + S_b(y, y, Ty) + S_b(z, z, Tz)), \quad \forall x, y, z \in \mathcal{X} \tag{2}$$

*where $B \in \mathcal{A}'_+$ and $\|B\| < \frac{1}{3}$ and $B \ne \frac{1}{3A}$. Then there exists a unique fixed point in $\mathcal{X}$.*

**Proof.** For convenience, and without loss of generality, we assume $B \ne 0$. Notice that $B \in \mathcal{A}'_+$, and $B(S_b(x, x, Tx) + S_b(y, y, Ty) + S_b(z, z, Tz))$ is also a positive element.

Choose $x_0 \in \mathcal{X}$ and set $x_{n+1} = Tx_n = T^{n+1}x_0$, $n = 1, 2, \ldots$. Let $B_0$ denote the element $d(x_1, x_0)$ in $\mathcal{A}$, and $S_{b_n} = S(x_n, x_n, x_{n+1})$.

$$\begin{aligned} S_{b_n} &= S_b(x_n, x_n, x_{n+1}) \\ &= S_b(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\le B(S_b(x_{n-1}, x_{n-1}, Tx_{n-1}) + S_b(x_{n-1}, x_{n-1}, Tx_{n-1}) + S_b(x_n, x_n, Tx_n)) \\ &= B(2S_b(x_{n-1}, x_{n-1}, Tx_{n-1}) + S_b(x_n, x_n, Tx_n)) \\ &= B(2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_n, x_n, x_{n+1})) \\ &= B(2S_{b_{n-1}} + S_{b_n}). \end{aligned}$$

Hence $(I-B)S_{b_n} \leq 2BS_{b_{n-1}}$. Thus

$$S_{b_n} \leq \frac{2B}{1-B} S_{b_{n-1}},$$

such that $\|B\| < \frac{1}{3}$. So

$$S_{b_n} \leq 2(I-B)^{-1} B S_{b_{n-1}} = t S_{b_{n-1}}.$$

Let $t = 2(I-B)^{-1}B$. By repeating this process, we obtain

$$S_{b_n} \leq t^n S_{b_0}.$$

Therefore, $\lim_{n\to\infty} S_{b_n} = 0$. Now, we prove that $\{x_n\}$ is a Cauchy sequence. It follows from (2) that for $n, m \in \mathbb{N}$

$$\begin{aligned} S_b(x_n, x_n, x_m) &= S_b(T^n x_0, T^n x_0, T^m x_0) \\ &= S_b(Tx_{n-1}, Tx_{n-1}, Tx_{m-1}) \\ &\leq B[2S_b(x_{n-1}, x_{n-1}, Tx_{n-1}) + S_b(x_{m-1}, x_{m-1}, Tx_{m-1})] \\ &= B[2S_b(x_{n-1}, x_{n-1}, x_n) + S_b(x_{m-1}, x_{m-1}, x_m)] \\ &= B(2S_{b_{n-1}} + S_{b_{m-1}}). \end{aligned}$$

Hence,

$$\|S_b(x_n, x_n x_m)\| \leq \|B\| \|2S_{b_{n-1}} + S_{b_{m-1}}\|.$$

Since $\lim_{n\to\infty} S_{b_n} = 0$, for every $\epsilon > 0$, we can find $N$ such that $\|S_{b_{n-1}}\| < \frac{\epsilon}{4}$ and $\|S_{b_{m-1}}\| < \frac{\epsilon}{2}$ for all $n, m > N$. Thus, we obtain $\|2S_{b_{n-1}} + S_{b_{m-1}}\| \leq 2\frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon$.

As $\|B\| < 1$ it follows that $\|S_b(x_n, x_n, x_m)\| < \epsilon$ for all $n, m > N$. Therefore, $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$ and $\lim_{n\to\infty} S_b(x_n, x_n, x_m) = 0$. Since $\mathcal{X}$ is complete, there exists $x \in \mathcal{X}$ such that

$$\lim_{n\to\infty} S_b(x_n, x_n, u) = \lim_{m,n\to\infty} (x_n, x_n, u) = S_b(u, u, u) = 0. \tag{3}$$

Now, we prove that $Tu = u$. For any $n \in \mathbb{N}$

$$\begin{aligned} S_b(u, u, Tu) &= A[S_b(u, u, x_{n+1}) + S_b(u, u, x_{n+1}) + S_b(Tu, Tu, x_{n+1})] \\ &= A[2S_b(u, u, x_{n+1}) + S_b(Tu, Tu, Tx_n)] \\ &\leq A[2S_b(u, u, x_{n+1}) + B(2S_b(u, u, Tu) + S_b(x_n, x_n, Tx_n))] \\ &= 2AS_b(u, u, x_{n+1}) + 2ABS_b(u, u, Tu) + ABS_b(x_n, x_n, Tx_n). \end{aligned}$$

Therefore,

$$(1 - 2AB)S_b(u, u, Tu) \leq 2AS_b(u, u, x_{n+1}) + ABS_b(x_n, x_n, Tx_n).$$

So

$$S_b(u, u, Tu) \leq \frac{2A}{(1-2AB)} S_b(u, u, x_{n+1}) + \frac{AB}{(1-2AB)} S_b(x_n, x_n, Tx_n).$$

Since $S_b(x_n, x_n, Tx_n) \to S_b(u, u, Tu)$ as $n \to \infty$, we obtain

$$\|S_b(u, u, Tu)\| \leq \|\frac{2A}{1-2AB}\| \|S_b(u, u, x_{n+1})\| + \|\frac{AB}{1-2AB}\| \|S_b(u, u, Tu)\|.$$

Therefore,

$$\|1 - \frac{AB}{1-2AB}\| \|S_b(u, u, Tu)\| \leq \|\frac{2A}{1-2AB}\| \|S_b(u, u, x_{n+1})\|,$$

and hence,

$$\|S_b(u, u, Tu)\| \leq \|\frac{2A}{1-3AB}\| \|S_b(u, u, x_{n+1})\| \to 0, \ (as\, n \to \infty).$$

Since $B \neq \frac{1}{3A}$ and from (3), we have $S_b(u, u, Tu) = 0$, which implies $Tu = u$, i.e., $u$ is a fixed point of $T$. Now suppose there exist $u, v \in \mathcal{X}$ such that $Tu = u$ and $Tv = v$. Then as (2), we get

$$\begin{aligned} S_b(u, u, v) &= S_b(Tu, Tu, Tv) \\ &\leq B[S_b(u, u, Tu) + S_b(u, u, Tu) + S_b(v, v, Tv)] \\ &= 2BS_b(u, u, Tu) + BS_b(v, v, Tv) \\ &= 2BS_b(u, u, u) + BS_b(v, v, v) \\ &= 0. \end{aligned}$$

Therefore, $u = v$. i.e., the fixed point of $T$ is unique. □

## 4. Application

First, consider the following example which will be needed in the next theorem.

**Example 4.1.** *Let $\mathcal{X} = L^\infty(E)$ and $H = L^2(E)$, where $E$ is Lebesque measurable set. Suppose $L(H)$ denote the set of bounded linear operators on hilbert space $H$. Clearly $L(H)$ is a $C^*$-algebra with usual operator norm.*

*Define $S_b : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to L(H)$ by $S_b(f, g, h) = \pi_{(|f-h|+|g-h|)^p}$, for all $f, g, h, \in \mathcal{X}$, where $\pi_h : H \to H$ is multiplication operator, $\pi_h(\varphi) = h.\varphi$ for $\varphi \in H$, and $p > 1$. Then $S_b$ is a $C^*$-algebra-valued $S_b$-metric and $(\mathcal{X}, L(H), S_b)$ is a complete $C^*$-algebra-valued $S_b$-metric space.*

*Let $\{f_n\}_{n=1}^\infty$ in $\mathcal{X}$ be a Cauchy sequence with respect to $L(H)$, i.e., for any $p \in \mathbb{N}$,*

$$\|S_b(f_{n+m}, f_{n+m}, f_n)\| \to 0, \ (n \to \infty).$$

*Now,*

$$\begin{aligned}\|S_b(f_{n+m}, f_{n+m}, f_n)\| &= \|\pi_{(|f_{n+m}-f_n|+|f_{n+m}-f_n|)^p}\| \\ &= \|\pi_{(2|f_{n+m}-f_n|)^p}\| \\ &= \|2(f_{n+m} - f_n)\|_\infty^p \to 0, \ (n \to \infty).\end{aligned}$$

*Then $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in the space $\mathcal{X}$. Since $\mathcal{X}$ is complete $C^*$-algebra-valued $S_b$-metric space, there exists $f \in \mathcal{X}$ such that*

$$\|f_n - f\|_\infty^p \to 0 \ (n \to \infty).$$

*Therefore,*

$$\begin{aligned}\|S_b(f_n, f_n, f)\| &= \|\pi_{(|f_n-f|+|f_n-f|)^p}\| \\ &= \|2(f_n - f)\|_\infty^p \\ &= 2^{p-1}\|(f_n - f)\|_\infty^p + 2^{p-1}\|(f_n - f)\|_\infty^p \to 0 \ (n \to \infty).\end{aligned}$$

*Hence the sequence $\{f_n\}_{n=1}^\infty$ converges to the function $f$ in $\mathcal{X}$ with respect to $L(H)$. Thus, $(\mathcal{X}, L(H), S_b)$ is complete with respect to $L(H)$.*

As application of contractive mapping theorem on complete $C^*$-algebra-valued $S_b$-metricspace, existence and uniqueness results for a type of integral equation and operator equation are given.

**Theorem 4.1.** *Consider the integral equation*

$$x(t) = \int_E K(t, s, x(s)) + g(t), \quad t \in E,$$

*where $E$ is the Lebesque measurable set. Assume that the following conditions hold:*

*(1) $K : E \times E \times \mathbb{R} \to \mathbb{R}$ is integrable and $g \in L^\infty(E)$.*

*(2) there exists a continuous function $\phi : E \times E \to \mathbb{R}$ and $k \in (0, 1)$ such that*

$$|K(t, s, u) - K(t, s, v)| \leq k|\phi(t, s)(u - v)|,$$

*for $t, s \in E$ and $u, v \in \mathbb{R}$.*

*(3) $\sup_{t\in E} \int_E |\phi(t, s)|ds \leq 1$.*

*Then the integral equation has a unique solution $x^*$ in $L^\infty(E)$.*

**Proof.** Let $\mathcal{X} = L^\infty(E)$ and $H = L^2(E)$, where $E$ is Lebesgue measurable set. Let $S_b$ be as defined in Example 4.1. Then $S_b$ is a $C^*$-algebra-valued $S_b$-metric and $(\mathcal{X}, L(H), S_b)$ is a complete $C^*$-algebra-valued $S_b$-metric space with respect to $L(H)$.

Define $T : L^\infty(E) \to L^\infty(E)$ by

$$Tx(t) = \int_E K(t, s, x(s))ds + g(t), \quad (t \in E).$$

Set $A = kI$. Then $A \in L(H)_+$ and $\|A\| = k < 1$. For any $h \in H$,

$$\begin{aligned}
\|S_b(Tx, Tx, Ty)\| &= \sup_{\|h\|=1} < \pi_{(|Tx-Ty|+|Tx-Ty|)^p} h, h > \\
&= \sup_{\|h\|=1} < \pi_{(2|Tx-Ty|)^p} h, h > \\
&= \sup_{\|h\|=1} < 2^p |Tx - Ty|^p h, h > \\
&= \sup_{\|h\|=1} \int_E (2^p |Tx - Ty|) h(t).\overline{h(t)} dt \\
&\leq 2^p \sup_{\|h\|=1} \int_E [\int_E |K(t, s, x(s)) - K(t, s, y(s))|]^p |h(t)|^2 dt \\
&\leq 2^p \sup_{\|h\|=1} \int_E [\int_E k|\phi(t, s)(x(s) - y(s))| ds]^p |h(t)|^2 dt \\
&\leq 2^p k^p \sup_{\|h\|=1} \int_E [\int_E |\phi(t, s)| ds]^p |h(t)|^2 dt. \|x - y\|_\infty^p \\
&\leq k \sup_{t\in E} \int_E |\phi(t, s)| ds. \sup_{\|h\|=1} \int_E |h(t)|^2 dt 2^p \|x - y\|_\infty^p \\
&\leq 2^p k \|x - y\|_\infty^p \\
&= k \|2(x - y)\|_\infty^p \\
&= k \|\pi_{(|x-y|+|x-y|)^p}\| \\
&= \|A\| \|S_b(x, x, y)\|.
\end{aligned}$$

Since $\|A\| < 1$, using Corollary 3.2, the integral equation has a unique solution in $L^\infty(E)$. ☐

## 5. Conclusions

In this paper, we study whether there are correspondence of some metric and fixed point properties in $S_b$-metric spaces taking the domain set of $S_b$-metric function in which $\mathcal{A}$ is a $C^*$-algebra-valued set. For this purpose, we first present $C^*$-algebra-valued $S_b$-metric space on the set having this structure by applying the properties of this algebraic concept. This specified structure is important in terms of integrating some metric constructions of fixed point theory and algebraic topology.

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Please cite this article using:

Seyede Samira Razavi, Hashem Parvaneh Masiha, $C^*$-algebra-valued $S_b$-metric spaces and applications to integral equations, AUT J. Math. Comput., 6(1) (2025) 31-39
https://doi.org/10.22060/AJMC.2023.22211.1140