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Original Article

Perfectness of the essential graph for modules over commutative rings

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ABSTRACT: Let R be a commutative ring and M be an R-module. The essential graph of M, denoted by $EG(M)$ is a simple graph with vertex set $Z(M) \setminus \mathrm{Ann}(M)$ and two distinct vertices $x, y \in Z(M) \setminus \mathrm{Ann}(M)$ are adjacent if and only if $\mathrm{Ann}_M(xy)$ is an essential submodule of M. In this paper, we investigate the dominating set, the clique and the chromatic number and the metric dimension of the essential graph for Noetherian modules. Let M be a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = n \geq 2$ and let $EG(M)$ be a connected graph. We prove that $EG(M)$ is a weakly prefect, that is, $\omega(EG(M)) = \chi(EG(M))$. Furthermore, it is shown that $\dim(EG(M)) = |Z(M)| - (|\mathrm{Ann}(M)| + 2^n)$, whenever $r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$ and $\dim(EG(M)) = |Z(M)| - (|\mathrm{Ann}(M)| + 2^n - 2)$, whenever $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$.

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1. Introduction

The study of algebraic structures by using the properties of a simple graph is a topic which becomes more attention in last decades and leads many authors to study and explore its properties. In fact, research on this subject aims is exposing the relationship between rings and modules theory with graphs theory, see for examples [1, 2]. Recently, the essential graph of a commutative ring was introduced and studied in [7]. Also, the concept of the essential graph for modules has been defined and studied in [10].

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For every connected vertices $u, v \in V(G)$, the distance between u and v is defined as the length of a shortest path from u to v and is denoted by $\mathrm{d}(u, v)$. We write $u \sim v$ if $\mathrm{d}(u, v) = 1$ and $u \nsim v$ otherwise. The degree of a vertex u, denoted by $\deg(u)$, is the number of edges incident to u. Assume that u is a vertex of G. The open neighborhood of u is defined as $N(u) = \{v \in V(G) : \mathrm{d}(u, v) = 1\}$ and the closed neighborhood of u is $N[u] = N(u) \cup \{u\}$. For distinct vertices $u, v \in V(G)$, if $N(u) = N(v)$, then u and v are non-adjacent twins. A clique of G is a complete subgraph of G and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the clique number of G. A dominating set of G is a subset D of $V(G)$ such that every vertex in $V(G) \setminus D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G

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is the minimum cardinality of a dominating set. The chromatic number of G, denoted by $\chi(G)$, is the minimal number of colors, which can assigned to the vertices of G in such a way that two adjacent vertices have different colors. The graph G is called weakly perfect whenever $\chi(G) = \omega(G)$. Let G be a connected graph. Assume that $W = \{w_1, w_2, \ldots, w_k\}$ is an ordered subset of $V(G)$. The metric representation of a vertex $u \in V(G)$ with respect to W is the vector $r(u|W) = (\mathrm{d}(u, w_1), \mathrm{d}(u, w_2), \ldots, \mathrm{d}(u, w_k))$. The set W called a resolving set for G if different vertices of G have different representation with respect to W. The minimum cardinality of any resolving set of G is the metric dimension of G and is denoted by $\dim(G)$, see [4].

Theorem 1.1 ([5, Corollary 2.4]). *Suppose that u, v are twins in a connected graph G and S resolves G. Then either u or v is in S. Moreover, if $u \in S$ and $v \notin S$, then $(S \setminus \{u\}) \cup \{v\}$ also resolves G.*

Let R be a commutative ring and let M be an R-module. The essential graph of M, denoted by $EG(M)$ is a simple graph with vertex set $Z(M) \setminus \mathrm{Ann}(M)$ and two distinct vertices $x, y \in Z(M) \setminus \mathrm{Ann}(M)$ are adjacent if and only if $\mathrm{Ann}_M(xy)$ is an essential submodule of M. The main goal of this paper is computing the domination, the clique and the chromatic number and the metric dimension of $EG(M)$. In section 3, we prove that if M is a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = n$ and $EG(M)$ is a connected graph, then the following statements are true:

(i) If $r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$, then $\dim(EG(M)) = |Z(M)| - (|\mathrm{Ann}(M)| + 2^n)$.
(ii) If $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$, then $\dim(EG(M)) = |Z(M)| - (|\mathrm{Ann}(M)| + 2^n - 2)$.

Throughout this paper, R is a commutative ring with non-zero identity and M is a unitary R-module. The set of zero-divisors of M, denoted by $Z(M)$ is defined to be the set $\{r \in R \ : \ rm = 0 \text{ for some } 0 \neq m \in M\}$. For $a \in R$, $\mathrm{Ann}_M(a) = \{m \in M \ : \ am = 0\}$. A proper submodule P of M is said to be prime whenever for $r \in R$ and $m \in M$, $rm \in P$ implies that $m \in P$ or $r \in \mathrm{Ann}_R(M/P)$. Let $\mathrm{Spec}_R(M)$ denote the set of prime submodules of M and $m - \mathrm{Ass}_R(M) = \{P \in \mathrm{Spec}_R(M) : P = \mathrm{Ann}_M(a) \text{ for some } 0 \neq a \in R\}$. For notations and terminologies not given in this article, the reader is referred to [9].

2. The Domination, Clique and Chromatic number of the Essential Graph

In this section we investigate the domination, clique and chromatic number of the essential graph. We at first, calculate the domination number for a Noetherian R-module M. The following theorem plays an important role in this paper, so for the convenience of the reader we write it here.

Theorem 2.1 ([10, Theorem 2.5]). *Let M be a Noetherian R-module with $\mathrm{Ann}(M) \neq r(\mathrm{Ann}(M))$. Then $x, y \in Z(M) \setminus \mathrm{Ann}(M)$ are adjacent in $EG(M)$ if and only if $xy \in \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{MinAss}_R(M)$.*

It is easy to see that the above theorem is true when $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$.

Lemma 2.2. *Let M be a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = 1$. Then the following conditions are equivalent:*

(i) *$r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$;*
(ii) *$EG(M)$ is a connected graph and $\gamma(EG(M)) = 1$.*

Proof. (i)⇒(ii) Suppose that $r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$. Then by [10, Lemma 2.2] each element of $r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)$ is a universal vertex of $EG(M)$. Hence, $EG(M)$ is a connected graph and for any $x \in r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)$, $D = \{x\}$ is a dominating set for $EG(M)$.

(ii)⇒(i) Suppose that $\mathrm{MinAss}_R(M) = \{\mathfrak{p}\}$ and $D = \{x\}$ is a dominating set for $EG(M)$. If $x \in r(\mathrm{Ann}(M))$, then there is nothing to prove. Otherwise, by the assumption there is $y \in Z(M) \setminus \mathrm{Ann}(M)$ such that $x \sim y$ is an edge of $EG(M)$. Now, $xy \in \mathfrak{p} = r(\mathrm{Ann}(M))$ which implies that $y \in r(\mathrm{Ann}(M))$. Hence, $r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$. Let $Z(M) = \mathrm{Ann}(M) \cup \{x\}$. Considering $x \neq x^2$ or $x = x^2$, we have $x^2 \in \mathrm{Ann}(M)$ or $1 - x \in \mathrm{Ann}(M)$, which are contradictions. □

Definition 2.3 ([8, Definition 2.1]). *Let M be an R-module. The zero-divisor graph of M, denoted by $\Gamma(M)$ is a simple undirected graph whose vertex set is $Z(M) \setminus \mathrm{Ann}(M)$ and two distinct vertices x and y are adjacent if and only if $xyM = 0$.*

If M is a Noetherian R-module with $|\mathrm{MinAss}_R(M)| = 1$ and $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$, then in view of [8, Lemma 2.1] and [10, Theorem 4.6], $EG(M)$ is an empty graph or it has only one vertex.

Let M be a Noetherian R-module and let $|\mathrm{MinAss}_R(M)| = n \geq 2$. If $r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$, then in view of Lemma 2.2, $\gamma(EG(M)) = 1$. In the following we consider the case $|\mathrm{MinAss}_R(M)| = n \geq 2$ with $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$.

Theorem 2.4. *Let M be a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = n \geq 2$ and let $EG(M)$ be a connected graph. Then $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$ if and only if $\gamma(EG(M)) = |\mathrm{MinAss}_R(M)|$.*

Proof. Suppose that

$$r(\mathrm{Ann}(M)) = \mathrm{Ann}(M), \quad \mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}, \quad m - \mathrm{Ass}(M) = \{P_1 = \mathrm{Ann}_M(x_1), \ldots, P_n = \mathrm{Ann}_M(x_n)\},$$

where $\mathfrak{p}_i = \mathrm{Ann}(M/P_i)$ and $x_i \in Z(M) \setminus \mathrm{Ann}(M)$, for all $1 \leq i \leq n$. Set $D = \{x_1, \ldots, x_n\}$. We show that D is a dominating set for $EG(M)$. Assume that $y \in Z(M) \setminus \mathrm{Ann}(M)$. Since $EG(M)$ is a connected graph by [10, Theorem 3.3], it follows that $\mathrm{Ann}_M(y) \not\subseteq \cap_{i=1}^{n} P_i$. Hence, there is i with $1 \leq i \leq n$ such that $\mathrm{Ann}_M(y) \not\subseteq P_i$. Therefore, $yx_iM = 0$ so y and x_i are adjacent, it follows from [8, Lemma 3.1(ii)]. Now, assume that $D' = \{x'_1, \ldots, x'_{n-1}\} \subseteq Z(M) \setminus \mathrm{Ann}(M)$. To prove the assertion, it is enough to show that D' is not a dominating set for $EG(M)$. Assume in contrary that D' is a dominating set for $EG(M)$ and we achieve a contradiction. By the hypothesis $\mathrm{Ann}(M) = r(\mathrm{Ann}(M)) = \cap_{i=1}^{n} \mathfrak{p}_i$ so for all $1 \leq j \leq n-1$ there exists $1 \leq i \leq n$ such that $x'_j \notin \mathfrak{p}_i$. Without loss of generality, we may assume that $x'_j \notin \mathfrak{p}_j$, for every $1 \leq j \leq n-1$. Suppose that $x \in \cap_{i=1}^{n-1} \mathfrak{p}_i \setminus \mathfrak{p}_n$ so $x \neq x'_j$, for all $1 \leq j \leq n-1$ and x is adjacent to x'_k for some $1 \leq k \leq n-1$. Then $xx'_kM = 0$ since by [10, Theorem 4.6] we have $EG(M) = \Gamma(M)$. So $xx'_k \in \cap_{i=1}^{n-1} \mathfrak{p}_i$ and hence $xx'_k \in \mathfrak{p}_k$, which is a contradiction. Therefore, D' is not a dominating set for $EG(M)$ and the proof is completed.
The converse is obvious by Lemma 2.2. □

Theorem 2.5. *Let M be a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = n \geq 2$ and let $EG(M)$ be a connected graph. Then*

$$\omega(EG(M)) = \begin{cases} n, & \text{if } r(\mathrm{Ann}(M)) = \mathrm{Ann}(M) \\ |r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)|, & \text{if } Z(M) = r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M) \\ |r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)| + n, & \text{if } Z(M) \neq r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M). \end{cases}$$

Proof. Suppose that $m - \mathrm{Ass}(M) = \{P_1 = \mathrm{Ann}_M(x_1), \ldots, P_n = \mathrm{Ann}_M(x_n)\}$, where $x_i \in Z(M) \setminus \mathrm{Ann}(M)$, for all $1 \leq i \leq n$ and $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$. Then by [10, Theorem 4.6] it follows that $\Gamma(M) = EG(M)$ and $X = \{x_1, \ldots, x_n\}$ is a clique for $EG(M)$ by [8, Lemma 3.1(i)]. Let $\Theta = \{\mathrm{Ann}_M(x) : x \in Z(M) \setminus \mathrm{Ann}(M)\}$. By hypotheses Θ has some maximal elements and by [6, Proposition 3.2] the maximal elements of Θ are prime submodules of M. Thus the set of maximal elements of Θ is a subset of $m - \mathrm{Ass}(M)$ so every element of Θ is a subset of at least one element of $m - \mathrm{Ass}(M)$. Assume $X' = \{x'_1, \ldots, x'_k\}$ $(k > n)$ is a maximal clique. Thus there are $1 \leq i \neq j \leq k$ such that $\mathrm{Ann}_M(x'_i), \mathrm{Ann}_M(x'_j) \subseteq \mathrm{Ann}_M(x_t)$, for some $1 \leq t \leq n$. Since x'_i and x'_j are adjacent so $x'_ix'_jM = 0$. Hence, $x'_iM \subseteq \mathrm{Ann}_M(x'_j) \subseteq \mathrm{Ann}_M(x_t)$. Therefore, $x_tx'_iM = 0$ so $x_tM \subseteq \mathrm{Ann}_M(x'_i)$. Thus $x_t^2M = 0$ which is a contradiction. Therefore, $\omega(EG(M)) = |\mathrm{MinAss}_R(M)|$.

Let $r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$ and $Z(M) = r(\mathrm{Ann}(M))$. Then by [10, Theorem 2.5], $EG(M)$ is a complete graph. Hence, $r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)$ is a clique and the result follows.

Now, assume that $Z(M) \neq r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$ and $\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. It is easy to see that $(r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)) \cup \{x_1, \ldots, x_n\}$ is a clique for $EG(M)$, where $x_j \in \cap_{i=1, i\neq j}^{n} \mathfrak{p}_i \setminus \mathfrak{p}_j$, for all $j = 1, \ldots, n$. Moreover, if $y \in Z(M) \setminus r(\mathrm{Ann}(M))$, then the set $r(\mathrm{Ann}(M) \setminus \mathrm{Ann}(M)) \cup \{x_1, \ldots, x_n, y\}$ is not a clique since for $y \in \cap_{i=1}^{t} \mathfrak{p}_i \setminus \cup_{i=t+1}^{n} \mathfrak{p}_i$ with $1 \leq t < n$ it is clear that $yx_{t+1} \notin r(\mathrm{Ann}(M))$ so y and x_{t+1} are not adjacent. Suppose that X is a clique for $EG(M)$. Thus in view of [10, Theorem 2.5], $(r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)) \subseteq X$. Let $X = (r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)) \cup \{y_1, \ldots, y_{n+1}\}$. Then there are $1 \leq i \neq j \leq k$ such that $y_i, y_j \notin \mathfrak{p}_t$, for some $1 \leq t \leq n$, but $y_iy_j \in r(\mathrm{Ann}(M)) \subseteq \mathfrak{p}_t$ which is a contradiction. Therefore, X is not a clique. □

Theorem 2.6. *Let M be a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = n \geq 2$ and let $EG(M)$ be a connected graph. Then*

$$\chi(EG(M)) = \begin{cases} |r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)|, & \text{if } Z(M) = r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M) \\ |r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)| + n, & \text{if } Z(M) \neq r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M) \\ n, & \text{if } r(\mathrm{Ann}(M)) = \mathrm{Ann}(M). \end{cases}$$

Proof. (i) In the case of $Z(M) = r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$, $EG(M)$ is a complete graph thus

$$\chi(EG(M)) = \omega(EG(M) = |r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)|$$

(ii) Assume that $Z(M) \neq r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$, $m - \mathrm{Ass}(M) = \{P_1 = \mathrm{Ann}_M(x_1), \ldots, P_n = \mathrm{Ann}_M(x_n)\}$, where $x_i \in Z(M) \setminus \mathrm{Ann}(M)$, for all $1 \leq i \leq n$ and $\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, where $\mathfrak{p}_i = \mathrm{Ann}(M/\mathrm{Ann}_M(x_i)) =$

$\mathrm{Ann}(x_iM)$, for all $1 \le i \le n$. By the proof of Theorem 2.5, $X = (r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)) \cup \{x_1, \ldots, x_n\}$ is a maximal clique for $EG(M)$ also $x_i \in \cap_{j=1, j\neq i}^{n} \mathfrak{p}_j \setminus \mathfrak{p}_i$, for all $i = 1, \ldots, n$. So

$$|r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)| + n \le \chi(EG(M)).$$

For each $y \in V(EG(M))$ with $y \notin (r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)) \cup \{x_1, \ldots, x_n\}$, let $\pi(y)$ denote the set $\{i : 1 \le i \le n, yx_i \notin \cap_{j=1}^{n} \mathfrak{p}_j\}$ thus it consist of indices $i \in \{1, \ldots, n\}$ such that two vertices y and x_i are not adjacent. Since the clique is maximal there exists at least one vertex $x_i \in \{x_1, \ldots, x_n\}$ such that y and x_i are not adjacent so $i \in \pi(y)$. Now we provide a proper vertex coloring for the graph $EG(M)$ with $|r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)| + n$ colors. For this purpose, color the vertices of this clique with $|r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)| + n$ different colors such that the color of vertices $x_1, \ldots, x_n$ be $1, \ldots, n$ respectively. Let $y \in V(EG(M)) \setminus (r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)) \cup \{x_1, \ldots, x_n\}$ and let the color of y be $\min\{j : j \in \pi(y)\}$. For each $j \in \pi(y)$ two vertices y and x_j are not adjacent, the color of y is different from the colors of vertices in the clique which are adjacent to y. Suppose that $y' \in V(EG(M)) \setminus (r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)) \cup \{x_1, \ldots, x_n\}$ is a vertex adjacent to y. We show that the colors of y and y' are different. Since y and y' are adjacent, we have $yy' \in \cap_{j=1}^{n} \mathfrak{p}_j$. Assume that $i = \min\{j : j \in \pi(y)\}$ so the color of y is i. By $x_i \in \cap_{j=1, j\neq i}^{n} \mathfrak{p}_j$ and $yx_i \notin \cap_{j=1}^{n} \mathfrak{p}_j$, we get that $y \notin \mathfrak{p}_i$. Also, $yy' \in \cap_{j=1}^{n} \mathfrak{p}_j$ implies that $yy' \in \mathfrak{p}_i$. Thus $y'x_i \in \cap_{j=1}^{n} \mathfrak{p}_j$ and hence $i \notin \pi(y')$. This implies that the color of y' is not i. This is a proper coloring and hence $\chi(EG(M)) \le |r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)| + n$. Therefore, $\chi(EG(M)) = |r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)| + n$.

(iii) The proof is similar to that of (ii). □

Recall that a graph G is said to be weakly perfect whenever $\omega(G) = \chi(G)$.

Corollary 2.7. *Let M be a Noetherian R-module and let $EG(M)$ be a connected graph. Then $EG(M)$ is a weakly perfect graph.*

A perfect graph G is a graph in which the chromatic number of every induced subgraph equals to the size of a largest clique of that subgraph. In 2006, M. Chudnovsky et al. settled a long standing conjecture regarding perfect graphs and provided a characterization of perfect graphs.

Theorem 2.8. [3, The Strong Perfect Graph Theorem] *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

In the following we investigate the perfectness of $EG(M)$.

Theorem 2.9. *Let M be a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = 1$. Then $EG(M)$ is perfect.*

Proof. If $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$, then in view of [8, Lemma 2.1] and [10, Theorem 4.6] it follows that $EG(M)$ is an empty graph or it has only one vertex. So neither $EG(M)$ nor $\overline{EG}(M)$ contains an induced odd cycle of length at least 5. Hence, Theorem 2.8 shows that $EG(M)$ is perfect. Now, let $r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$ and

$$x_1 \sim x_2 \sim \ldots \sim x_k \sim x_1$$

be an induced odd cycle of length $k \ge 5$ in $EG(M)$. Then $x_1x_2 \in r(\mathrm{Ann}(M))$ which implies that either $x_1 \in r(\mathrm{Ann}(M))$ or $x_2 \in r(\mathrm{Ann}(M))$. Hence, by [10, Lemma 2.2] either $x_1 \sim x_2 \sim x_3 \sim x_1$ or $x_2 \sim x_3 \sim x_4 \sim x_2$ is a cycle which is a contradiction. Suppose that

$$y_1 \sim y_2 \sim \ldots \sim y_k \sim y_1$$

is an induced odd cycle of length $k \ge 5$ in $\overline{EG}(M)$. Since y_2 and y_3 are adjacent vertices in $\overline{EG}(M)$ so $y_2y_3 \notin r(\mathrm{Ann}(M))$ on the other hand since y_1 and y_3 are adjacent vertices in $EG(M)$ thus $y_1y_3 \in r(\mathrm{Ann}(M))$, so $y_1 \in r(\mathrm{Ann}(M))$. Thus by [10, Lemma 2.2], y_1 is a universal vertex of $EG(M)$ and so is a single vertex in $\overline{EG}(M)$, that is a contradiction. Hence, neither $EG(M)$ nor $\overline{EG}(M)$ contains an induced odd cycle of length at least 5. Therefore, $EG(M)$ is a perfect graph by Theorem 2.8. □

Theorem 2.10. *Let M be a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = 2$. Then $EG(M)$ is perfect.*

Proof. Note that if $x \in r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)$, then, by [10, Lemma 2.2], x is a universal vertex of $EG(M)$ and so is a single vertex in $\overline{EG}(M)$. Thus x can not be a vertex in an induced cycle of length at least 5 in $EG(M)$ and $\overline{EG}(M)$. Assume that $\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ and

$$x_1 \sim x_2 \sim \ldots \sim x_k \sim x_1$$

is an induced odd cycle of length $k \geq 5$ in $EG(M)$. Thus $x_1x_2 \in r(\mathrm{Ann}(M)) = \mathfrak{p}_1 \cap \mathfrak{p}_2$. Without loss of generality we may assume that $x_1 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$. Thus $x_2 \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Hence, from $x_2x_3 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ it follows that $x_3 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ and from $x_3x_4 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ it follows that $x_4 \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. These facts show that $x_1 \sim x_4$ is an edge of $EG(M)$ which is a contradiction. Suppose that

$$y_1 \sim y_2 \sim \ldots \sim y_k \sim y_1$$

is an induced odd cycle of length $k \geq 5$ in $\overline{EG}(M)$. Since $y_1 \sim y_3$ is an edge in $EG(M)$ so

$$y_1y_3 \in r(\mathrm{Ann}(M)) = \mathfrak{p}_1 \cap \mathfrak{p}_2.$$

Without loss of generality we may assume that $y_1 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$. Thus $y_3 \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Also, from $y_1y_4 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ it follows that either $y_4 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ or $y_4 \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. If $y_4 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$, then $y_3 \sim y_4$ is an edge in $EG(M)$ which is a contradiction. So $y_4 \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Hence, from $y_2y_4 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ it follows that $y_2 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ and from $y_2y_5 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ it follows that $y_5 \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Now, $y_1 \sim y_5$ is an edge of $EG(M)$ that is a contradiction. So neither $EG(M)$ nor $\overline{EG}(M)$ contains an induced odd cycle of length at least 5. Therefore, $EG(M)$ is a perfect graph by Theorem 2.8. □

Theorem 2.11. *Let M be a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = 3$. Then $EG(M)$ is perfect.*

Proof. Assume that $\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ we show that $EG(M)$ not contains an induced odd cycle of length at least 5. Suppose that

$$x_1 \sim x_2 \sim \ldots \sim x_k \sim x_1$$

is an induced odd cycle of length $k \geq 5$ in $EG(M)$. As in the proof of Theorem 2.10, $x_i \notin r(\mathrm{Ann}(M))$, for all $1 \leq i \leq k$. From $x_1x_2 \in r(\mathrm{Ann}(M)) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$ and $x_1x_k \in r(\mathrm{Ann}(M)) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$ it follows that:
Case 1. $x_1 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2 \cup \mathfrak{p}_3$. So $x_2, x_k \in \mathfrak{p}_2 \cap \mathfrak{p}_3 \setminus \mathfrak{p}_1$. Hence, $x_{k-1} \in \mathfrak{p}_1$ since $x_{k-1}x_k \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$. Now, $x_2 \sim x_{k-1}$ is an edge of $EG(M)$ which is a contradiction.
Case 2. $x_1 \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \setminus \mathfrak{p}_3$. So $x_2 \in \mathfrak{p}_3$ or $x_2 \in \mathfrak{p}_1 \cap \mathfrak{p}_3$ or $x_2 \in \mathfrak{p}_2 \cap \mathfrak{p}_3$ and $x_k \in \mathfrak{p}_3$ or $x_k \in \mathfrak{p}_1 \cap \mathfrak{p}_3$ or $x_k \in \mathfrak{p}_2 \cap \mathfrak{p}_3$. If $x_2 \in \mathfrak{p}_3$ and $x_k \in \mathfrak{p}_3$, then $x_{k-1} \in \mathfrak{p}_1 \cap \mathfrak{p}_2$. Hence, $x_2 \sim x_{k-1}$ which is a contradiction. If $x_2 \in \mathfrak{p}_3$ and $x_k \in \mathfrak{p}_1 \cap \mathfrak{p}_3$, then $x_3 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ and $x_{k-1} \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ or $x_{k-1} \in \mathfrak{p}_2 \cap \mathfrak{p}_3$ or $x_{k-1} \in \mathfrak{p}_2$. In the first case for x_{k-1}, $x_2 \sim x_{k-1}$ is an edge of $EG(M)$ and in second case $x_1 \sim x_{k-1}$ is an edge of $EG(M)$ that are contradiction. In the third case for x_{k-1} we have $x_{k-2} \in \mathfrak{p}_1 \cap \mathfrak{p}_3$ which shows that $x_1 \sim x_{k-2}$ is an edge is a contradiction. In other cases, by similar arguments we achieve to contradiction. Therefore, $EG(M)$ not contains an induced odd cycle of length at least 5. Suppose that

$$y_1 \sim y_2 \sim \ldots \sim y_k \sim y_1$$

is an induced odd cycle of length $k \geq 5$ in $\overline{EG}(M)$. Since $y_1 \sim y_3$ is an edge of $EG(M)$ so $y_1y_3 \in r(\mathrm{Ann}(M)) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$ now it follows that:
Case 1. $y_1 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2 \cup \mathfrak{p}_3$. Thus $y_3 \in \mathfrak{p}_2 \cap \mathfrak{p}_3 \setminus \mathfrak{p}_1$. Since $y_1 \sim y_4$ and $y_2 \sim y_4$ are two edges in $EG(M)$ it follows that $y_4 \in \mathfrak{p}_2 \cap \mathfrak{p}_3$ and $y_2 \in \mathfrak{p}_1$. Now, $y_2 \sim y_3$ is an edge of $EG(M)$ which is a contradiction.
Case 2. $y_1 \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \setminus \mathfrak{p}_3$. So $y_3 \in \mathfrak{p}_3$ or $y_3 \in \mathfrak{p}_1 \cap \mathfrak{p}_3$ or $y_3 \in \mathfrak{p}_2 \cap \mathfrak{p}_3$ and $y_4 \in \mathfrak{p}_3$ or $y_4 \in \mathfrak{p}_1 \cap \mathfrak{p}_3$ or $y_4 \in \mathfrak{p}_2 \cap \mathfrak{p}_3$. If $y_3 \in \mathfrak{p}_3$ and $y_4 \in \mathfrak{p}_3$, then $y_2 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$. Hence, $y_2 \sim y_3$ is an edge of $EG(M)$ which is a contradiction. If $y_3 \in \mathfrak{p}_3$ and $y_4 \in \mathfrak{p}_1 \cap \mathfrak{p}_3$, then $y_2 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ or $y_2 \in \mathfrak{p}_2 \cap \mathfrak{p}_3$ or $y_2 \in \mathfrak{p}_2$. In the first case for y_2, $y_2 \sim y_3$ is an edge of $EG(M)$ and in second case $y_1 \sim y_2$ is an edge of $EG(M)$ that are contradiction. Now, assume that $y_2 \in \mathfrak{p}_2$, so from $y_3y_5 \in r(\mathrm{Ann}(M))$ we have $y_5 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ thus $y_4 \sim y_5$ is an edge of $EG(M)$ that is a contradiction. In other cases, by similar arguments we achieve to contradiction. Hence, $\overline{EG}(M)$ not contains an induced odd cycle of length at least 5. Therefore, $EG(M)$ is a perfect graph by Theorem 2.8. □

Remark 2.12. *Although the graph $EG(M)$ is perfect for $|\mathrm{MinAss}_R(M)| \leq 3$ but for an R-module M with*

$$\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}, \quad (n \geq 4)$$

it is not perfect. Since $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_5 \sim x_1$ is a cycle of length 5 in $EG(M)$, where $x_1 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, $x_2 \in \mathfrak{p}_1 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4 \setminus \mathfrak{p}_2$, $x_3 \in \mathfrak{p}_2 \cap \mathfrak{p}_3 \setminus \mathfrak{p}_1 \cap \mathfrak{p}_4$, $x_4 \in \mathfrak{p}_1 \cap \mathfrak{p}_4 \setminus \mathfrak{p}_2 \cap \mathfrak{p}_3$ and $x_5 \in \mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4 \setminus \mathfrak{p}_1$.

3. Metric Dimension of the Essential Graph

In this section we calculate the metric dimension of the essential graph for modules over commutative ring R.

Notation 3.1. *For $k \in \mathbb{N}$ with $1 \leq k \leq n$, we shall write*

$$\mathcal{I}(k, n) := \{(i(1), \ldots, i(k)) \in \mathbb{N}^k : 1 \leq i(1) < i(2) < \ldots < i(k) \leq n\},$$

the set of all strictly increasing sequences of length k of positive integers taken from the set $\{1, \ldots, n\}$. For $i \in \mathcal{I}(k,n)$, we shall, for $1 \leq j \leq k$, denote the j-th component of i by $i(j)$, so that $i = (i(1), \ldots, i(k))$. Suppose that $k < n$ and $i \in \mathcal{I}(k,n)$. By the n-complement of i we mean the sequence $\bar{i} \in \mathcal{I}(n-k,n)$ such that $\{1, \ldots, n\} = \{i(1), \ldots, i(k), \bar{i}(1), \ldots, \bar{i}(n-k)\}$.

Theorem 3.2. *Let M be a Noetherian R-module such that $|\mathrm{MinAss}_R(M)| = n$. Then the following statements are true:*

(i) *If $r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$, then $\dim(EG(M)) = |Z(M)| - (|\mathrm{Ann}(M)| + 2^n)$.*
(ii) *If $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$ and $EG(M)$ is a connected graph, then*

$$\dim(EG(M)) = |Z(M)| - (|\mathrm{Ann}(M)| + 2^n - 2).$$

Proof. (i) Suppose that $\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $r(\mathrm{Ann}(M)) \neq \mathrm{Ann}(M)$. Then $EG(M)$ is a connected graph since each vertex of $r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M)$ is universal. It is straightforward to see that $Z(M) \setminus \mathrm{Ann}(M)$ separates into 2^n disjoint subsets such as $A_1 = Z(M) \setminus \cup_{i=1}^n \mathfrak{p}_i$ and $A_i = \mathfrak{p}_{i(1)} \cap \ldots \cap \mathfrak{p}_{i(k)} \setminus \mathfrak{p}_{\bar{i}(1)} \cup \ldots \cup \mathfrak{p}_{\bar{i}(n-k)}$, where $i \in \mathcal{I}(k,n)$ and $1 \leq k \leq n$. It is easy to see that for $i \in \mathcal{I}(k,n)$, $1 \leq k \leq n$ and $x_i, x'_i \in A_i$, $N(x_i) = N(x'_i) = \mathfrak{p}_{\bar{i}(1)} \cap \ldots \cap \mathfrak{p}_{\bar{i}(n-k)}$, see Theorem 2.1. Assume that W is a resolving set for $EG(M)$ so by Theorem 1.1, $A_i \setminus \{x_i\} \subseteq W$. Hence, $Z(M) \setminus (\mathrm{Ann}(M) \cup \{x_1, \ldots, x_{2^n}\}) \subseteq W$. We know that x_i and x_j have different neighbours, for all $1 \leq i \neq j \leq 2^n$, so one can easily show $x_1, \ldots, x_{2^n}$ have different coordinates with respect to W. Therefore, $W = Z(M) \setminus (\mathrm{Ann}(M) \cup \{x_1, \ldots, x_{2^n}\})$ is a resolving set for $EG(M)$. If $\emptyset = A_1 = Z(M) \setminus \cup_{i=1}^n \mathfrak{p}_i$, then $W = Z(M) \setminus (\mathrm{Ann}(M) \cup \{x_2, \ldots, x_{2^n}\})$.
(ii) Suppose that $\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Since $EG(M)$ is a connected graph, $A_1 = Z(M) \setminus \cup_{i=1}^n \mathfrak{p}_i$ is an empty set, by Theorem 2.1. On the hand, $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$ so $Z(M) \setminus \mathrm{Ann}(M)$ separates into $2^n - 2$ disjoint subsets. Now, by a similar argument to that of (i) one can show that $W = Z(M) \setminus (\mathrm{Ann}(M) \cup \{x_2, \ldots, x_{2^n-1}\})$ is a resolving set for $EG(M)$. □

We end this section with the following examples that related to previous results.

Example 3.1. *It is easy to see that any element of a finite ring is a unit or a zero divisor. So for a positive integer n we have $|Z(\mathbb{Z}_n)| = n - \phi(n)$ and $|Z(\mathbb{Z}_n)^*| = n - 1 - \phi(n)$, where ϕ is the Euler phi function.*

(i) *If p is a prime number, then $EG(\mathbb{Z}_p) = \Gamma(\mathbb{Z}_p)$ is an empty graph.*
(ii) *If $n = p^\alpha$ for some prime number p and an integer $\alpha \geq 2$, then $Z(EG(\mathbb{Z}_n))^* = \mathrm{Nil}(\mathbb{Z}_n)^*$ so by [10, Theorem 2.10], $EG(\mathbb{Z}_n)$ is a complete graph. Thus $Z(EG(\mathbb{Z}_n))^* = (p\mathbb{Z}_n)^*$ is a maximal clique so $\omega(EG(\mathbb{Z}_n)) = \chi(EG(\mathbb{Z}_n)) = n - 1 - \phi(n) = p^\alpha - 1 - p^\alpha + p^{\alpha-1} = p^{\alpha-1} - 1$. Moreover, $D = \{p\}$ is a dominating set for $EG(\mathbb{Z}_n)$ so $\gamma(EG(\mathbb{Z}_n)) = 1$. Also, $\dim(EG(\mathbb{Z}_n)) = p^{\alpha-1} - 2$.*
Consider the ring $\mathbb{Z}_{16}$. It is clear that $\mathrm{Ass}(\mathbb{Z}_{16}) = \{2\mathbb{Z}_{16}\}$ and $Z(\mathbb{Z}_{16}) = \mathrm{Nil}(\mathbb{Z}_{16}) = 2\mathbb{Z}_{16}$. Thus $EG(\mathbb{Z}_{16})$ is a complete graph with 7 vertices and $D = \{2\}$ is a dominating set for it and $\dim(EG(\mathbb{Z}_{16})) = 2^3 - 2 = 6$.

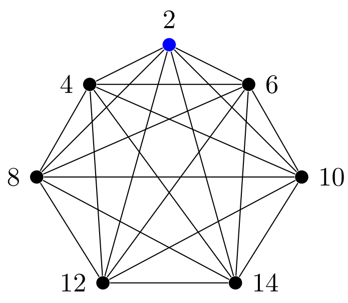

Figure 1: A complete graph with 7 vertices.

(iii) *If $n = p_1 \ldots p_k$ $(k \geq 2)$ for some distinct prime number p_i, then $Z(EG(\mathbb{Z}_n)) = p_1\mathbb{Z}_n \cup \ldots \cup p_k\mathbb{Z}_n$ and $\mathrm{Nil}(\mathbb{Z}_n) = \{0\}$. $X = \{\prod_{i=1, i\neq j}^k p_i : j = 1, \ldots, k\}$ is a maximal clique hence $\omega(EG(\mathbb{Z}_n)) = \chi(EG(\mathbb{Z}_n)) = k$. $D = \{\prod_{i=1, i\neq j}^k p_i : j = 1, \ldots, k\}$ is a dominating set for $EG(\mathbb{Z}_n)$ so $\gamma(EG(\mathbb{Z}_n)) = k$. Moreover,*

$$\dim(EG(\mathbb{Z}_n)) = |Z(\mathbb{Z}_n)| - (|\mathrm{Ann}(\mathbb{Z}_n)| + 2^k - 2) = n - \phi(n) - 2^k + 1.$$

Consider the ring $\mathbb{Z}_{15}$. It is clear that $\mathrm{Ass}(\mathbb{Z}_{15}) = \{3\mathbb{Z}_{15}, 5\mathbb{Z}_{15}\}$ and $\mathrm{Nil}(\mathbb{Z}_{15}) = 3\mathbb{Z}_{15} \cap 5\mathbb{Z}_{15} = 0$. By [10, Theorem 3.7], $EG(\mathbb{Z}_{15}) = K_{2,4}$ the complete bipartite graph with 8 vertices and $D = \{3, 5\}$ is a dominating set for $EG(\mathbb{Z}_{15})$.

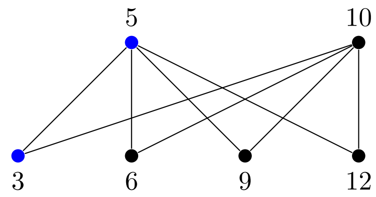

Figure 2: A complete bipartite graph with 6 vertices.

(iv) *If $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ $(k \geq 2)$, where p_i is a prime number and α_i is a positive integer for all $1 \leq i \leq k$, $X = (p_1 \dots p_k\mathbb{Z}_n)^* \cup \{p_1, p_2, \dots, p_k\}$ is a maximal clique hence $\omega(EG(\mathbb{Z}_n)) = \chi(EG(\mathbb{Z}_n)) = p_1^{\alpha_1 - 1} \dots p_k^{\alpha_k - 1} + k$. $D = \{p_i\}$ is a dominating set for $EG(\mathbb{Z}_n)$ so $\gamma(EG(\mathbb{Z}_n)) = 1$. Moreover,*

$$\dim(EG(\mathbb{Z}_n)) = |Z(\mathbb{Z}_n)| - (|\operatorname{Ann}(\mathbb{Z}_n)| + 2^k - 1) = n - \phi(n) - 2^k.$$

Consider the ring $\mathbb{Z}_{12}$. It is clear that $\operatorname{Ass}(\mathbb{Z}_{12}) = \{2\mathbb{Z}_{12}, 3\mathbb{Z}_{12}\}$, $\operatorname{Nil}(\mathbb{Z}_{12}) = 2\mathbb{Z}_{12} \cap 3\mathbb{Z}_{12} = 6\mathbb{Z}_{12}$ and $D = \{6\}$ is a dominating set for $EG(\mathbb{Z}_{12})$.

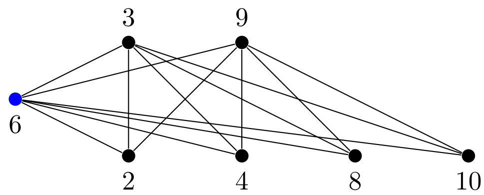

Figure 3: A graph with 7 vertices.

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