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Original Article

Numerical solutions of fractional Fokker-Planck equation with hybrid method of solution

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ABSTRACT: The work employs a numerical method for the solution of Fractional Fokker-Planck Equation (FFPE) using the Homotopy Perturbation and Aboodh Transform Method (HPATM). Fractional derivatives issues are successfully solved using the hybrid approach, which yields rapidly convergent solutions. By resolving two cases and contrasting estimated outcomes with exact solutions for various fractional orders, the correctness of the technique was proven. The accuracy of the technique is demonstrated by the good match between the precise and approximation solutions at $\alpha = 1$. The findings indicate that fractional differential equations may be solved with a strong and dependable approach using HPATM, which can also be used to describe anomalous diffusion and other intricate physical phenomena.

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1. Introduction

A basic model utilized in many scientific domains, such as biology, economics, physics, and more, is the Fractional Fokker-Planck Differential Equation (FFPE) [12]. FFPE is unique in its ability to describe the emergence of complex systems under stochastic processes and fractional calculus principles [19]. However, solving FFPE analytically poses significant challenges due to its nonlinearity and the presence of fractional-order derivatives.

For modeling complex systems, the FFPE is an effective mathematical tool, especially when long-range interdependence and anomalous diffusion are present [8, 10]. Its adaptability across a variety of disciplines, including physics, economics, and biology, is well acknowledged [1]. However, due of their intrinsic nonlinearity and the fractional character of the related derivatives, FFPEs are particularly challenging to solve analytically.

When dealing with FFPEs, traditional analytic techniques for differential equations frequently encounter difficulties, especially when attempting to adequately handle the fractional order and nonlinearity. Alternative methods

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have been investigated in response to this gap, however semi-analytic techniques like the Fractional Variational Iteration Method (FVIM) [6, 13, 17], Fractional Differential Transform Method (FDTM) [7], ZZ transform [16], Extended Kudryashov Method [23], and Homotopy Perturbation Method (HPM) [5, 9] also has it complexities.

Frequently encountered problems encompass convergence concerns, computing inefficiencies, and challenges in handling highly nonlinear situations or intricate beginning conditions.

In an effort to close this knowledge gap, this work presents the time-Fractional Fokker-Planck Equation and the Homotopy Perturbation Aboodh Transform Method (HPATM) as a useful method for solving nonlinear fractional differential equations. Homotopy Perturbation Method and Aboodh Transform strengths are combined in HPATM to provide a more powerful and adaptable framework. In order to overcome the shortcomings of conventional approaches, this hybrid approach increases convergence rates and boosts solution accuracy.

The Fokker-Planck equation provides a mathematical description of the probability density function (PDF) of a stochastic process that takes into account both random fluctuations and deterministic forces. [20] provides this equation's generic form as:

$$\frac{\partial \varphi(z,\tau)}{\partial \tau} = -\frac{\partial}{\partial z}\left[Q(z)\varphi(z,\tau)\right] + \frac{\partial^2}{\partial z^2}\left[R(z)\varphi(z,\tau)\right],$$

where:

- $\varphi(z,\tau)$ represents the PDF of the system at position z and time τ,
- $Q(z)$ is the drift coefficient, representing the deterministic force acting on the system,
- $R(z)$ is the diffusion coefficient, representing the strength of stochastic fluctuations.

If the order of the equation is fractional, say of order σ, then the equation is referred to as the Fractional Fokker-Planck Equation (FFPE). Furthermore, the equation is nonlinear if Q or R is a function of z, τ, or φ; otherwise, it remains linear.

The FFPE can be expressed as:

$$D_t^{\sigma}\varphi(z,\tau) = \left[-\frac{\partial}{\partial z}Q(z,\tau,\varphi) + \frac{\partial^2}{\partial z^2}R(z,\tau,\varphi)\right]\varphi(z,\tau), \qquad 0 < \sigma < 1,$$

with

$$\varphi(z,0) = h(z), \qquad z \in \Re.$$

The Homotopy Perturbation Aboodh Transform Method (HPATM) has emerged as a potent technique for tackling these complex issues [2, 3, 4, 11, 14, 15, 18, 21]. An study of a generic fractional partial differential equation (FPDE) applying the initial conditions serves to illustrate the fundamental ideas of this approach.

In particular, the time-Fractional Fokker-Planck Equation is studied, together with the mathematical basis of the Homotopy Perturbation and Aboodh Transform Method (HPATM) for solving nonlinear differential equations. The format of the paper is as follows: The components and methodology of HPATM are presented in Section 2, along with a description of its creation and guiding ideas. In Section 3, HPATM is applied to two particular FFPE samples, and its efficacy is demonstrated by graphical analysis and numerical data. The study is finally concluded in Section 4, which offers a thorough examination of HPATM and its role in the numerical solution of fractional differential equations.

2. Materials and Method

This section illustrates the algorithms for the given method for solving Fractional Fokker-Planck. We give the necessary definitions and Theorems that will make the work much easier.

2.1. Derivative of Two Parameter Mittag-Leffler Function MLF

The Aboodh transform's inversion formula is found using the two-parameter Mittag-Leffler method. Thus,

$$\frac{d}{dt}E_{\sigma,\sigma}(\tau) = E'_{\sigma,\sigma} = \sum_{j=1}^{\infty}\frac{j\tau^{j-1}}{\Gamma(\sigma j + \sigma)}.$$

Letting $j-1=r$ implies $j=r+1$, and we have

$$E'_{\sigma,\sigma} = \sum_{r=0}^{\infty}\frac{(r+1)\tau^{r}}{\Gamma(\sigma r + \sigma + \sigma)}.$$

Further differentiation gives,

$$D''E_{\sigma,\sigma}(\tau) = \sum_{r=1}^{\infty} \frac{r(r+1)\tau^{r-1}}{\Gamma(\sigma r + \sigma + \sigma)},$$

and letting $r-1=j$ implies $r=j+1$ and we have (note that n,k are dummy variables)

$$D''E_{\sigma,\sigma}(\tau) = \sum_{j=0}^{\infty} \frac{(j+1)(j+2)\tau^{j}}{\Gamma(\sigma j + 2\sigma + \sigma)}.$$

Moreover, by the property of the binomial coefficient

$$(r+1) = \frac{(r+1)!}{r!}, \quad (r+1)(r+2) = \frac{(r+2)!}{r!}, \quad \dots \quad , \frac{(r+k)!}{r!} = \frac{r!}{(r-m)!},$$

we observe that the m-th derivative is

$$D^m E_{\sigma,\sigma} = E^{(m)}_{\sigma,\sigma} = \sum_{r=m}^{\infty} \frac{r!\tau^{r-m}}{(r-m)!\Gamma(\sigma r + \sigma)}. \tag{1}$$

Letting $r-m=j$ implies $r=j+m$, then equation (1) becomes

$$D^m E_{\sigma,\sigma}(\pm a\tau^{\sigma})^j = E^{(m)}_{\sigma,\sigma} = \sum_{j=0}^{\infty} \frac{(j+m)!(\pm a\tau^{\sigma})^j}{j!\Gamma(\sigma j + \sigma m + \sigma)}.$$

Consider the approximate solution of the form

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{(j+m)!z^j}{j!} &= \sum_{j=0}^{\infty}(m+j)\dots(j+1)z^j \\ &= \sum_{j=m}^{\infty} j(j-1)(j-2)\dots(j-m+1)z^{j-m} \\ &= D^m \sum_{j=0}^{\infty} z^j = D^m\left[\frac{1}{1-z}\right] = \frac{m!}{(1-z)^{m+1}}.\end{aligned}$$

Therefore,

$$\sum_{j=0}^{\infty} \frac{(j+m)!z^j}{j!} = \frac{m!}{(1-z)^{m+1}}.$$

2.2. Aboodh Transform of Mittag-Leffler Funtions

Lemma 2.1. *For $\sigma > 0$ and $v^{\sigma} > |b|$, we have the following inverse Aboodh transform [18]*

$$\mathcal{A}^{-1}\left[\frac{v^{\sigma-(1+\sigma)}}{v^{\sigma \pm b}}\right] = [\tau^{\sigma-1}E_{\sigma,\sigma}(\mp b\tau^{\sigma})].$$

Proof. We have

$$\begin{aligned}\mathcal{A}[\tau^{\sigma-1}E_{\sigma,\sigma}(\mp b\tau^{\sigma})] &= \frac{1}{v}\int_0^{\infty} e^{-vt}\tau^{\sigma-1}\sum_{j=0}^{\infty}\frac{(\pm b\tau^{\sigma})^j}{\Gamma(\sigma k+\sigma)} \\ &= \frac{1}{v}\int_0^{\infty} e^{-vt}\tau^{\sigma-1}.\tau^{\sigma k}\sum_{j=0}^{\infty}\frac{(\pm b)^j}{\Gamma(\sigma k+\sigma)}dt \\ &= \sum_{j=0}^{\infty}\frac{(\pm b)^j}{\Gamma(\sigma k+\sigma)}\frac{1}{v}\int_0^{\infty} e^{-vt}\tau^{\sigma-1+\sigma k}dt,\end{aligned}$$

where

$$\frac{1}{v}\int_0^{\infty} e^{-vt}\tau^{\sigma-1+\sigma k}dt = \mathcal{A}[\tau^{\sigma k+\sigma-1}].$$

So,

$$\begin{aligned}\mathcal{A}[\tau^{\sigma-1}E_{\sigma,\sigma}(\mp b\tau^{\sigma})] &= \sum_{j=0}^{\infty}\frac{(\pm b)^j}{\Gamma(\sigma k+\sigma)},\mathcal{A}[\tau^{\sigma k+\sigma-1}]\\ &= \sum_{j=0}^{\infty}\frac{(\pm b)^j}{\Gamma(\sigma k+\sigma)},\frac{\Gamma(\sigma k+\sigma)}{v^{\sigma k+\sigma+1}}\\ &= \sum_{j=0}^{\infty}(\pm b)^j v^{(-\sigma k-\sigma-1)} = v^{-(1+\sigma)}\sum_{j=0}^{\infty}(\pm b)^j v^{(-\sigma k)}.\end{aligned}$$

Using the identity,

$$\sum_{r=0}^{\infty}(\pm z)^r = \frac{1}{1\mp z},$$

we obtain

$$\mathcal{A}[\tau^{\sigma-1}E_{\sigma,\sigma}(\mp b\tau^{\sigma})] = v^{-(1+\sigma)}.\frac{1}{1\pm bv^{-\sigma}} = \frac{v^{-(1+\sigma)}}{1\mp bv^{-\sigma}} = \frac{v^{\sigma-(1+\sigma)}}{v^{\sigma}\pm b}. \quad \square$$

Table 1: Aboodh transforms of some MLF [18, 21]

S/N	$K(v)$	$y(\tau)=\mathcal{A}^{-1}\{y(j)\}$
1.	$\frac{1}{v^{\sigma+1}}$	$\frac{\tau^{\sigma-1}}{\Gamma(\sigma)}$
2.	$\frac{1}{v(v^{\sigma}-a)}$	$\tau^{\sigma-1}E_{\sigma,\sigma}(a\tau^{\sigma})$
3.	$\frac{v^{\sigma-2}}{(v^{\sigma}+a)}$	$E_{\sigma}(-\sigma\tau^{\sigma})$
4.	$\frac{a}{v^2(v^{\sigma}+a)}$	$1-E_{\sigma}(-\sigma\tau^{\sigma})$
5.	$\frac{v^{-(\sigma+1)}}{(v-a)}$	$\tau^{\sigma}E_{1,\sigma+1}(at)$
6.	$\frac{v^{\sigma-(1+\sigma)}}{v^{\sigma}-a}$	$\tau^{\sigma-1}E_{\sigma,\sigma}(a\tau^{\sigma})$
7.	$\frac{v^{\sigma-(1+\sigma)}}{(v-a)^{\sigma}}$	$\frac{\tau^{\sigma-1}}{\Gamma(\sigma)}F_1(\sigma;\sigma;at)$
8.	$\frac{1}{v(v-a)(v-b)}$	$\frac{1}{a-b}(e^{at}-e^{bt})$

2.3. Homotopy Perturbation Aboodh Transform Method (HPATM)

The non-homogeneous equation of the following generic nonlinear fractional PDE is taken into consideration:

$$D_t^{\sigma}\varphi(z,\tau) = L(\varphi(z,\tau)) + N(\varphi(z,\tau)) + f(z,\tau), \quad \sigma>0, \tag{2}$$

with

$$\varphi(z,0)=c_k, \quad j=0,\ldots,r-1, \quad D_0^r\varphi(z,0)=0 \quad \text{and} \quad r=[\sigma], \tag{3}$$

where

D_t^{σ}	denotes fractional derivative operator
f	represents the source term
N	represents the nonlinear term
L	represents the linear term.

Using the Aboodh Transform to solve equation (2). Thus, we get

$$\mathcal{A}[D_t^{\sigma}\varphi(z,\tau)] = \mathcal{A}[L(\varphi(z,\tau))] + \mathcal{A}[N(\varphi(z,\tau))] + \mathcal{A}[f(z,\tau)].$$

Applying Aboodh differential property and substituting equation (3), we have

$$\mathcal{A}[D_t^{\sigma}\varphi(z,\tau)] = \frac{1}{v^{\sigma}}\mathcal{A}[L(\varphi(z,\tau))] + \frac{1}{v^{\sigma}}\mathcal{A}[N(\varphi(z,\tau))] + g(z,\tau). \tag{4}$$

Solving equation (4) involves taking the inverse Aboodh transforms of both sides.

$$\varphi(z,\tau) = G(z,\tau) + \mathcal{A}^{-1}\left[\frac{1}{v^{\sigma}}\mathcal{A}[L(\varphi(z,\tau))] + \frac{1}{v^{\sigma}}\mathcal{A}[N(\varphi(z,\tau))]\right]. \tag{5}$$

Now, we apply HPM to equation (5) to get

$$\varphi(z,\tau) = \sum_{r=0}^{\infty} \varrho^r \varphi_r(z,\tau), \tag{6}$$

the nonlinear terms are expressed as

$$N\varphi(z,\tau) = \sum_{r=0}^{\infty} \varrho^r H_r(\varphi), \tag{7}$$

where $H_r(\varphi)$ are He's polynomial and given by

$$H_r(\varphi_0,\varphi_1,\varphi_2,\varphi_3,\dots,\varphi_r) = \frac{1}{r!}\frac{\partial^r}{\partial \varrho^r}\left[N\left(\sum_{i=0}^{\infty}\varrho^i\varphi_i(z,\tau)\right)\right]_{\varrho=0}, \qquad (r=0,1,2,\dots).$$

Combining equations (6) and (7) with equation (5) yields

$$\sum_{r=0}^{\infty}\varrho^r\varphi_r(z,\tau) = G(z,\tau) + \varrho\left[\mathcal{A}^{-1}\left[\frac{1}{v^{\sigma}}\mathcal{A}[L(\sum_{r=0}^{\infty}\varrho^r\varphi_r(z,\tau))] + \frac{1}{v^{\sigma}}\mathcal{A}[N(\sum_{r=0}^{\infty}\varrho^r\varphi_r(z,\tau))]\right]\right].$$

This is the use of He's polynomials to couple the Homotopy Perturbation Method with the Aboodh Transform. When we compare ϱ's coefficient of comparable powers, we get

$$\begin{aligned}
\varrho^0 &: \varphi_0(z,\tau) = Gz(z,\tau),\\
\varrho^1 &: \varphi_1(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^{\sigma}}\mathcal{A}\{-H_0 + H_0^*\}\right],\\
\varrho^2 &: \varphi_2(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^{\sigma}}\mathcal{A}\{-H_1 + H_1^*\}\right],\\
\varrho^3 &: \varphi_3(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^{\sigma}}\mathcal{A}\{-H_2 + H_2^*\}\right],\\
&\vdots\\
\varrho^r &: \varphi_r(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^{\sigma}}\mathcal{A}\{-H_{r-1} + H_{r-1}^*\}\right].
\end{aligned}$$

Therefore, the equation (2) may be solved as follows:

$$\varphi(z,\tau) = \varphi_0(z,\tau) + \varphi_1(z,\tau) + \varphi_2(z,\tau) + \varphi_3(z,\tau) + \cdots.$$

2.4. Fractional Fokker-Planck Differential Equation using HPATM

This section discusses the general Fokker-Planck equation and the HPATM technique is used to solve the nonlinear space-time fractional Fokker-Planck equations (FFPE),

$$D_t^{\sigma}\varphi(z,\tau) = \left[-\frac{\partial}{\partial z}Q(z,\tau,\varphi) + \frac{\partial^2}{\partial z^2}R(z,\tau,\varphi)\right]\varphi(z,\tau), \quad \tau>0, \quad z>0, \quad 0<\sigma\le 1, \tag{8}$$

with

$$\varphi(z,0) = f(z).$$

The Aboodh transform to equation (8) gives

$$\mathcal{A}[\varphi(z,\tau)] = \frac{1}{v^2}[f(z)] + \frac{1}{v^{\sigma}}\left[\mathcal{A}\left[-\frac{\partial}{\partial z}Q(z,\tau,\varphi)\varphi(z,\tau) + \frac{\partial^2}{\partial z^2}R(z,\tau,\varphi)\varphi(z,\tau)\right]\right].$$

Taking the Aboodh Inverse Transform, we obtain

$$\varphi(z,\tau) = \frac{1}{v^2}[f(z)] + \mathcal{A}^{-1}\left[\frac{1}{v^\sigma}\left[\mathcal{A}\left[-\frac{\partial}{\partial z}\sum_{r=0}^{\infty} Q_r + \frac{\partial^2}{\partial z^2}\sum_{r=0}^{\infty} R_r\right]\right]\right],$$

where

$$\sum_{r=0}^{\infty} Q_r = Q(z,\tau,\varphi)\varphi(z,\tau),$$

$$\sum_{r=0}^{\infty} R_r = R(z,\tau,\varphi)\varphi(z,\tau),$$

Applying Homotopy perturbation method, yields

$$\sum_{r=0}^{\infty} \varrho^r v_r(z,\tau) = g(z) + \varrho\left[\mathcal{A}^{-1}\left[\frac{1}{v^\sigma}\left[\mathcal{A}\left[-\sum_{r=0}^{\infty}\varrho^r H_r(z,\tau,\varphi) + \sum_{r=0}^{\infty}\varrho^r H_r^*(z,\tau,\varphi)\right]\right]\right]\right].$$

The He's polynomials $H_r(z,\tau,\varphi)$ and $H_r^*(z,\tau,\varphi)$ represent nonlinear terms, described by

$$\sum_{r=0}^{\infty} H_r(z,\tau,\varphi) = \frac{\partial}{\partial z}\left(\sum_{r=0}^{\infty} Q_r\right),$$

$$\sum_{r=0}^{\infty} H_r^*(z,\tau,\varphi) = \frac{\partial^2}{\partial z^2}\left(\sum_{r=0}^{\infty} R_r\right).$$

The coefficient of the corresponding power of ϱ can be collected using the following equations.

$$\begin{aligned}
\varrho^0 &: \varphi_0(z,\tau) = f(z),\\
\varrho^1 &: \varphi_1(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^\sigma}\mathcal{A}\{-H_0 + H_0^*\}\right],\\
\varrho^2 &: \varphi_2(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^\sigma}\mathcal{A}\{-H_1 + H_1^*\}\right],\\
\varrho^3 &: \varphi_3(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^\sigma}\mathcal{A}\{-H_2 + H_2^*\}\right],\\
&\vdots\\
\varrho^r &: \varphi_r(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^\sigma}\mathcal{A}\{-H_{r-1} + H_{r-1}^*\}\right].
\end{aligned}$$

The result of equation (8) is

$$\varphi(z,\tau) = \varphi_0(z,\tau) + \varphi_1(z,\tau) + \varphi_2(z,\tau) + \cdots + \varphi_r(z,\tau),$$

$$\varphi(z,\tau) = \lim_{r\to\infty}\sum_{r=0}^{N}\varphi_r(z,\tau).$$

2.5. Convergence Analysis of the method

To solve the convergence analysis of the HPATM, we would generally follow the procedures below:

i. The HPATM typically expresses the solution as a series expansion:

$$\varphi(z,\tau) = \varphi_0(z,\tau) + p\varphi_1(z,\tau) + p^2\varphi_2(z,\tau) + \cdots,$$

where p is an embedding parameter that varies from 0 to 1, and $\varphi_0(z,\tau), \varphi_1(z,\tau), \varphi_2(z,\tau), \ldots$ are the components of the solution series.

ii. A homotopy that bridges the gap between the actual problem and a simple problem that we can solve perfectly is created using the Homotopy Perturbation Method (HPM):

$$H(v,p) = (1-p)L(v) + pN(v) = 0,$$

where the embedding parameter is p, the linear operator is represented by $L(v)$ and the nonlinear operator by $N(v)$. The simplified problem arises when $p = 0$, whereas the original difficulty appears when $p = 1$..

iii. Substituting the series expansion of $\varphi(z,\tau)$ into the homotopy equation:

$$H(\varphi_0 + p\varphi_1 + p^2\varphi_2 + \cdots, p) = (1-p)L(\varphi_0 + p\varphi_1 + p^2\varphi_2 + \cdots) + pN(\varphi_0 + p\varphi_1 + p^2\varphi_2 + \cdots) = 0.$$

This yields a series of equations by equating coefficients of like powers of p. These equations are solved iteratively to determine $\varphi_0(z,\tau), \varphi_1(z,\tau), \varphi_2(z,\tau), \ldots$.

iv. The error after truncating the series at the n-th term is given by:

$$R_r(z,\tau) = \varphi(z,\tau) - \sum_{k=0}^{n} \varphi_k(z,\tau).$$

For the method to be convergent, we require that $\lim_{n\to\infty} R_r(z,\tau) = 0$. This means the sum of the infinite series must converge to the exact solution $\varphi(z,\tau)$.

v. Radius of Convergence
The convergence of the series depends on the radius of convergence, which is the value of p for which the series sum remains finite. In the case of HPATM, we need this radius to include $p = 1$, as the solution is evaluated at $p = 1$.

vi. A sufficient condition for convergence in many cases is that the norm of the rth term $\varphi_r(z,\tau)$ decreases exponentially:

$$\|\varphi_r(z,\tau)\| \leq C\rho^r \quad \text{for some} \quad 0 < \rho < 1,$$

where C is a constant, and ρ is a convergence factor. If this condition is met, the series converges for $p = 1$, meaning the HPATM produces a convergent solution.

3. Application

In order to assess the correctness of HPATM approaches and investigate the effects of altering the sequence of space- and time-fractional derivatives on solution behavior, this section provides Examples of such techniques.

Example 3.1. Examine the nonlinear time-fractional FPE of the form [16]

$$D_t^{\sigma}\varphi(z,\tau) = \left[-\frac{\partial}{\partial z}(z) + \frac{\partial^2}{\partial z^2}\left(\frac{z^2}{2}\right)\right]\varphi(z,\tau), \quad \tau \geq 0, \quad z \geq 0, \quad 0 < \sigma \leq 1, \tag{9}$$

with

$$\varphi(z,0) = z.$$

Taking the Aboodh Transform of equation (9), and applying the initial condition we obtain

$$v^{\sigma}\mathcal{A}[\varphi(z,\tau)] = \frac{1}{v^{\sigma+2}}\{z\} + \mathcal{A}\left[-\frac{\partial}{\partial z}(z)\varphi(z,\tau) + \frac{\partial^2}{\partial z^2}\left(\frac{z^2}{2}\right)\varphi(z,\tau)\right],$$

and so

$$\mathcal{A}[\varphi(z,\tau)] = \frac{1}{v^2}\{z\} + \frac{1}{v^{\sigma}}\mathcal{A}\left[-\frac{\partial}{\partial z}(z)\varphi(z,\tau) + \frac{\partial^2}{\partial z^2}\left(\frac{z^2}{2}\right)\varphi(z,\tau)\right]. \tag{10}$$

Now, the Aboodh Inverse Transform of equation (10) yields

$$\varphi(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^2}\{z\}\right] + \mathcal{A}^{-1}\left[\frac{1}{v^{\sigma}}\mathcal{A}\left[-\frac{\partial}{\partial z}(z)\varphi(z,\tau) + \frac{\partial^2}{\partial z^2}\left(\frac{z^2}{2}\right)\varphi(z,\tau)\right]\right],$$

and so

$$\varphi(z,\tau) = z + \mathcal{A}^{-1}\left[\frac{1}{v^{\sigma}}\mathcal{A}\left[-\frac{\partial}{\partial z}(z)\varphi(z,\tau) + \frac{\partial^2}{\partial z^2}\left(\frac{z^2}{2}\right)\varphi(z,\tau)\right]\right],$$

with $\varphi_0(z,\tau) = z$. Then,

$$\begin{aligned}
\varrho^0 &: \varphi_0(z,\tau) = z,\\
\varrho^1 &: \varphi_1(z,\tau) = \frac{z\tau^{\sigma}}{\Gamma(\sigma+1)},\\
\varrho^2 &: \varphi_2(z,\tau) = \frac{z\tau^{2\sigma}}{\Gamma(2\sigma+1)},\\
\varrho^3 &: \varphi_3(z,\tau) = \frac{z\tau^{3\sigma}}{\Gamma(3\sigma+1)},\\
\varrho^4 &: \varphi_4(z,\tau) = \frac{z\tau^{4\sigma}}{\Gamma(4\sigma+1)},\\
\varrho^5 &: \varphi_5(z,\tau) = \frac{z\tau^{5\sigma}}{\Gamma(5\sigma+1)}.
\end{aligned}$$

Hence,

$$\varphi(z,\tau) = z\left(1 + \frac{\tau^{\sigma}}{\Gamma(\sigma+1)} + \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)} + \frac{\tau^{5\sigma}}{\Gamma(5\sigma+1)} + \cdots\right). \tag{11}$$

That is,

$$\varphi(z,\tau) = zE_{\sigma}(\tau^{\sigma}).$$

The solution obtained is in good agreement with the exact solution obtained in [16]

If $\sigma = 1$ and using the property of the Gamma function $\Gamma(r+1) = r!$, we obtain the exact solution in classical calculus as

$$\varphi(z,\tau) = ze^{\tau}.$$

This result correlates with the result obtained in [16]

Table 2: Numerical solution of equation (11) with different values of σ and it exact solution

t	z	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$	$\sigma = 1$	Exact
0	0	0	0	0	0	0	0
0.1	0.1	0.352927384	0.202773137	0.148561314	0.125558955	0.110517092	0.110517092
0.2	0.2	0.792765327	0.497891048	0.358382008	0.291829536	0.244280552	0.244280552
0.3	0.3	1.278214002	0.865730398	0.625993940	0.499989339	0.404957642	0.404957642
0.4	0.4	1.797406550	1.300143451	0.953582488	0.754203381	0.596729879	0.596729879
0.5	0.5	2.344040833	1.797837072	1.344423041	1.059625210	0.824360635	0.824360635
0.6	0.6	2.914103715	2.356732885	1.802192545	1.422040217	1.093271280	1.093271280
0.7	0.7	3.504776913	2.975398572	2.330744937	1.847733520	1.409626895	1.409626895
0.8	0.8	4.113954145	3.652790943	2.934017024	2.343425719	1.780432743	1.780432743
0.9	0.9	4.739991543	4.388120941	3.615983236	2.916235990	2.213642800	2.213642800
1	1	5.381564434	5.180775281	4.380631945	3.573658298	2.718281828	2.718281828

Example 3.2. *Examine the nonlinear time-fractional FPE of the form [16, 20]*

$$D_t^{\sigma}\varphi(z,\tau) = \left[-\frac{\partial}{\partial z}\left(\frac{4\varphi}{z} - \frac{z}{3}\right) + \frac{\partial^2}{\partial z^2}(\varphi)\right]\varphi(z,\tau), \quad \tau \geq 0, \quad z \geq 0, \quad 0 < \sigma \leq 1, \tag{12}$$

with the initial condition

$$\varphi(z,0) = z^2.$$

Applying Aboodh transform to both sides, and applying the initial condition, we have

$$v^{\sigma}\mathcal{A}[\varphi(z,\tau)] = \frac{1}{v^{\sigma+2}}\{z^2\} + \mathcal{A}\left[-\frac{\partial}{\partial z}\left(\frac{4\varphi}{z} - \frac{z}{3}\right)\varphi(z,\tau) + \frac{\partial^2}{\partial z^2}(\varphi)\varphi(z,\tau)\right],$$

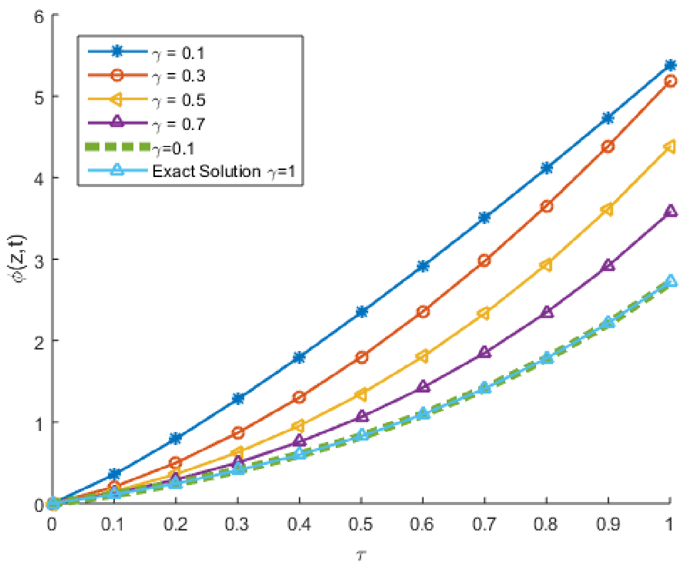

Figure 1: Behaviour of the solution at $\sigma = 0.1$, $\sigma = 0.3$, $\sigma = 0.5$, $\sigma = 0.7$ $\sigma = 1$, and the exact solution

Table 3: Comparison with other methods

t	z	ZZ transform[16]	HPM [22]	HPATM	Exact
0.1	0.1	0.110517092	0.110517092	0.110517092	0.110517092
0.2	0.2	0.244280552	0.244280552	0.244280552	0.244280552
0.3	0.3	0.404957642	0.404957642	0.404957642	0.404957642
0.4	0.4	0.596729879	0.596729879	0.596729879	0.596729879
0.5	0.5	0.824360635	0.824360635	0.824360635	0.824360635
0.6	0.6	1.093271280	1.093271280	1.093271280	1.093271280
0.7	0.7	1.409626895	1.409626895	1.409626895	1.409626895
0.8	0.8	1.780432743	1.780432743	1.780432743	1.780432743
0.9	0.9	2.213642800	2.213642800	2.213642800	2.213642800
1	1	2.718281828	2.718281828	2.718281828	2.718281828

and then

$$\mathcal{A}[\varphi(z,\tau)] = \frac{1}{v^2}\{z^2\} + \frac{1}{v^\sigma}\mathcal{A}\left[-\frac{\partial}{\partial z}\left(\frac{4\varphi}{z} - \frac{z}{3}\right)\varphi(z,\tau) + \frac{\partial^2}{\partial z^2}(\varphi)\varphi(z,\tau)\right].$$

Applying the Aboodh Inverse Transform $\mathcal{A}^{-1}$, we have

$$\varphi(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v^2}\{z\}\right] + \mathcal{A}^{-1}\left[\frac{1}{v^\sigma}\mathcal{A}\left[-\frac{\partial}{\partial z}\left(\frac{4\varphi}{z} - \frac{z}{3}\right)\varphi(z,\tau) + \frac{\partial^2}{\partial z^2}(\varphi)\varphi(z,\tau)\right]\right],$$

and so

$$\varphi(z,\tau) = z^2 + \mathcal{A}^{-1}\left[\frac{1}{v^\sigma}\mathcal{A}\left[-\frac{\partial}{\partial z}\left(\frac{4\varphi}{z} - \frac{z}{3}\right)\varphi(z,\tau) + \frac{\partial^2}{\partial z^2}(\varphi)\varphi(z,\tau)\right]\right],$$

where $\varphi_0(z,\tau) = z^2$.

Using the HPM discussed in section 2, we obtain the coefficients of powers of ϱ

$$\begin{aligned}
\varrho^0 &: \varphi_0(z,\tau) = z^2,\\
\varrho^1 &: \varphi_1(z,\tau) = \frac{z^2\tau^{\sigma}}{\Gamma(\sigma+1)},\\
\varrho^2 &: \varphi_2(z,\tau) = \frac{z^2\tau^{2\sigma}}{\Gamma(2\sigma+1)},\\
\varrho^3 &: \varphi_3(z,\tau) = \frac{z^2\tau^{3\sigma}}{\Gamma(3\sigma+1)},\\
\varrho^4 &: \varphi_4(z,\tau) = \frac{z^2\tau^{4\sigma}}{\Gamma(4\sigma+1)},\\
\varrho^5 &: \varphi_5(z,\tau) = \frac{z^2\tau^{5\sigma}}{\Gamma(5\sigma+1)}.
\end{aligned}$$

Hence,

$$\varphi(z,\tau) = z^2\left(1+\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}+\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+\frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}+\frac{\tau^{5\sigma}}{\Gamma(5\sigma+1)}+\cdots\right). \tag{13}$$

That is,

$$\varphi(z,\tau) = z^2E_{\sigma}(\tau^{\sigma}). \tag{14}$$

Equation (13) and (14) are the approximate and and exact solutions of equation (12) which are in good agreement. If we set $\sigma = 1$ and using the property of the Gamma function $\Gamma(r+1) = r!$. We obtain the exact solution

$$\varphi(z,\tau) = z^2e^{\tau}. \tag{15}$$

This result correlates with the result obtained in [16, 18, 20]

Table 4: Numerical solution of equation (13) with different values of σ and it exact solution

t	z	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$	$\sigma = 1$	Exact
0	0	0	0	0	0	0	0
0.1	0.1	0.035292738	0.020277314	0.014856131	0.012555895	0.011051708	0.011051709
0.2	0.2	0.158553065	0.099578210	0.071676402	0.058365907	0.048856000	0.048856110
0.3	0.3	0.383464200	0.259719119	0.187798182	0.149996802	0.121485375	0.121487293
0.4	0.4	0.718962620	0.520057380	0.381432995	0.301681352	0.238677333	0.238691952
0.5	0.5	1.172020416	0.898918536	0.672211520	0.529812605	0.412109375	0.412180318
0.6	0.6	1.748462229	1.414039731	1.081315527	0.853224130	0.655704000	0.655962768
0.7	0.7	2.453343839	2.082779001	1.631521456	1.293413464	0.985963708	0.986738827
0.8	0.8	3.291163316	2.922232754	2.347213619	1.874740575	1.422336000	1.424346194
0.9	0.9	4.265992389	3.949308847	3.254384913	2.624612391	1.987608375	1.992278520
1	1	5.381564434	5.180775281	4.380631945	3.573658298	2.708333333	2.718281828

4. Conclusion

The study employs the Homotopy Perturbation Aboodh Transform Method (HPATM) to solve a Fokker-Planck time-fractional problem. By utilizing He's polynomial, the nonlinear component of the equation is linearized, significantly reducing its complexity. The Aboodh transform is then applied to construct the fractional Fokker-Planck equation (FFPE). The behavior of the approximate solution is analyzed for different values of σ, and several numerical examples are provided to demonstrate its effectiveness. When σ is set as an integer, the exact solution within the realm of classical calculus is obtained. The study concludes that HPATM is a reliable and effective method for solving nonlinear differential equations, requiring only a limited number of terms for convergence. Numerical simulations conducted in MATLAB confirm the method's accuracy. Furthermore, when compared to other studies, this method is found to be highly accurate.

Table 5: Comparison with other methods

t	z	ILTM [20]	HPM [22]	HPATM	Exact
0.1	0.1	0.011051708	0.011051708	0.011051708	0.011051709
0.2	0.2	0.048856000	0.048856000	0.048856000	0.048856110
0.3	0.3	0.121485375	0.121485375	0.121485375	0.121487293
0.4	0.4	0.238677333	0.238677333	0.238677333	0.238691952
0.5	0.5	0.412109375	0.412109375	0.412109375	0.412180318
0.6	0.6	0.655704000	0.655704000	0.655704000	0.655962768
0.7	0.7	0.985963708	0.985963708	0.985963708	0.986738827
0.8	0.8	1.422336000	1.422336000	1.422336000	1.424346194
0.9	0.9	1.987608375	1.987608375	1.987608375	1.992278520
1	1	2.708333333	2.708333333	2.708333333	2.718281828

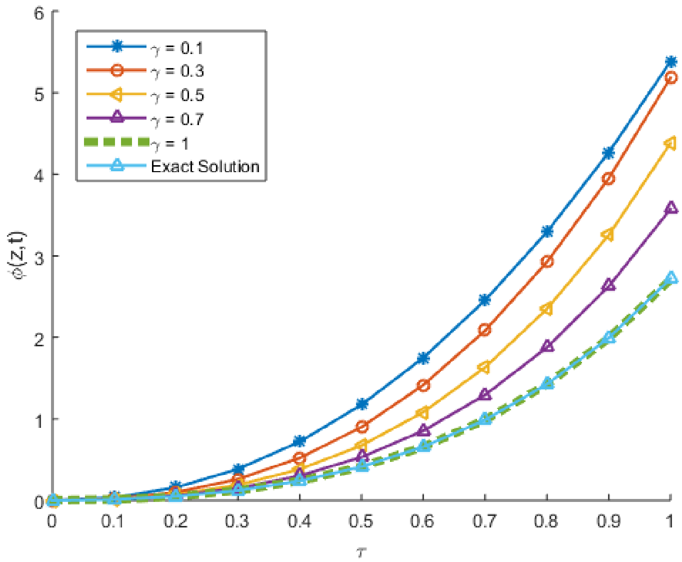

Figure 2: Behaviour of the solution at $\sigma = 0.1$, $\sigma = 0.3$, $\sigma = 0.5$, $\sigma = 0.7$ $\sigma = 1$, and the exact solution

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