AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 7(2) (2026) 183-188
https://doi.org/10.22060/AJMC.2024.23677.1285

Original Article

Almost complex structure over almost contact metric structures

Akbar Sadighi[*a], Seyyed Mohammad Zamanzadeh[b]

[a] *Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran*
[b] *Department of Mathematics, Bijar Branch, Islamic Azad University, Bijar, Iran*

ABSTRACT: In this paper, we investigate the conditions under which a lifted almost complex structure J on the tangent bundle TM of a manifold M exhibits various Kählerian properties. We establish several characterizations relating the geometry of (TM, J) to the cosymplectic structure on M. Specifically, we show that (TM, J) is Kählerian if and only if (M, η, ξ, φ) is cosymplectic and $R = 0$. Similarly, we prove that (TM, J) is nearly Kählerian under the same conditions on M. Furthermore, we present an alternative criterion for (TM, J) to be Kählerian, involving a nearly cosymplectic condition on M alongside a specific curvature relation. Finally, we demonstrate that (TM, J) is semi-Kählerian if and only if (M, η, ξ, φ) is semi-cosymplectic with $R(X, Y)\varphi Z = 0$. These results reveal intricate connections between cosymplectic structures on M and Kählerian-type structures on TM, contributing to the broader understanding of almost complex geometry on tangent bundles.

Review History:

Received:15 November 2024
Revised:05 January 2024
Accepted:22 December 2024
Available Online:01 May 2026

Keywords:

Almost complex structure
Cosymplectic structure
Nearly cosymplectic structures
Kählerian manifolds

MSC (2020):

53B40; 53C60

1. Introduction

The study of almost complex and almost contact metric structures forms a central theme in differential geometry, particularly due to their applications in both mathematical physics and complex geometry. An almost complex structure provides a generalization of complex structures on manifolds, enabling the extension of complex analytic techniques to real differentiable manifolds. In a similar vein, almost contact metric structures, which are defined on odd-dimensional manifolds, offer a natural framework for examining various geometric properties that do not necessarily require a complex structure.

An almost contact metric structure consists of a triple (ϕ, ξ, η, g), where ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form, and g is a Riemannian metric. This structure provides a rich setting for studying geometric aspects of manifolds that exhibit symmetries analogous to those in complex geometry. Almost complex structures on the other hand, involve an endomorphism J on even-dimensional manifolds that satisfies $J^2 = I$, where I is the

[*] *Corresponding author.*
E-mail addresses: sadighi@iaut.ac.ir (A. Sadighi), zamanzadeh.mohammad@gmail.com (S. M. Zamanzadeh)

identity map. These two structures can be combined in certain ways, allowing for new and interesting interactions to emerge in the geometry of odd- and even-dimensional manifolds.

In recent years, researchers have focused on understanding the interplay between almost complex and almost contact structures in various contexts, particularly in cases where almost complex structures can be constructed over the almost contact metric structures of an underlying manifold. Such interactions often lead to the discovery of new geometric invariants, integrability conditions, and curvature properties. Exploring these aspects not only enriches our understanding of the individual structures but also provides insights into how complex geometry might generalize in contexts that do not meet the usual integrability conditions.

In this paper, we investigate the properties of almost complex structures defined over almost contact metric structures. Specifically, we explore conditions under which the almost complex structure interacts harmoniously with the almost contact metric structure and examine the implications of these interactions for the curvature and topology of the underlying manifold. We also present several examples to illustrate our findings and discuss potential applications of these results in broader geometric contexts.

Our main results present significant characterizations of almost complex structures lifted to the tangent bundle $TM$and their relationships with various types of cosymplectic structures on the base manifold M. Here's a brief overview of each result and its implications:

First Theorem: This theorem establishes that the tangent bundle (TM, J) acquires a Kählerian structure precisely when the base manifold (M, η, ξ, φ) is cosymplectic and the curvature tensor R vanishes. This result suggests that under these conditions, the lifted structure exhibits the rich geometric properties associated with Kähler manifolds.

Second Theorem: Here, the nearly Kählerian condition on (TM, J) aligns with (M, η, ξ, φ) being cosymplectic with zero curvature. The theorem implies that the nearly Kähler condition, which relaxes the integrability constraint of the Kähler structure, still demands a flat and cosymplectic structure on the base manifold.

Third Theorem: This result provides an alternative criterion for (TM, J) to be Kählerian, involving a "nearly cosymplectic" condition on M along with a specific condition on the action of R relative to η. This characterization emphasizes a more intricate relationship between the curvature and the cosymplectic structure, broadening the scope of Kähler conditions on the lifted structure.

Fourth Theorem: For (TM, J) to be semi-Kählerian, the base manifold M must be semi-cosymplectic, and the curvature $R(X, Y)$ must satisfy $R(X, Y)\varphi Z = 0$. This theorem highlights a partial integrability condition on the curvature, providing a nuanced criterion for the semi-Kählerian property in terms of the base manifold's structure.

These results not only deepen the understanding of lifted almost complex structures but also uncover new connections between cosymplectic, nearly cosymplectic, and semi-cosymplectic structures. They pave the way for exploring further geometric and topological properties of the lifted manifold (TM, J) in relation to the structure of (M, η, ξ, φ).

2. Preliminaries

Let M be a differentiable manifold of dimension $2n + 1$. Suppose η, ξ and φ be a $1-$form, a vector field and a $(1, 1)-$tensor respectively. (η, ξ, φ) is called an almost contact structure on M if satisfies the following conditions:

$$\varphi(\xi) = 0, \quad \eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi.$$

A differentiable manifold of odd dimension with an almost contact structure is called an almost contact manifold. If a manifold M with the (η, ξ, φ) structure admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

then M is called an almost contact metric structure and g is called compatible metric. We can define two-form as follows

$$\Phi(X, Y) = g(X, \varphi(Y)),$$

for all $X, Y \in \chi(M)$. An almost contact structure (η, ξ, φ) is said to be normal if the almost complex structure J on $M \times \mathbb{R}$ given by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

where f is a C^∞ function on $M \times \mathbb{R}$ is integrable, which is equivalent to the condition $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ which can define as follows:

$$[\varphi, \varphi](X, Y) = [X, Y] + \varphi[\varphi X, Y] + \varphi[X, \varphi Y] - [\varphi X, \varphi Y] + d\eta(X, Y)\xi.$$

An almost contact metric structure (η, ξ, φ) in M is said to be:

1. Almost cosymplectic if $d\Phi = 0$ and $d\eta = 0$;
2. Cosymplectic if it is almost cosymplectic and normal;
3. Semi cosymplectic if $\delta\Phi = 0$ and $\delta\eta = 0$;
4. Nearly cosymplectic if $(\nabla_X \varphi)(Y) + (\nabla_Y \varphi)(X) = 0$ and $\nabla\eta = 0$.

An almost complex manifold M is a differentiable manifold equipped with a $(1, 1)$ tensor J which satisfies $J^2 = -I$, where I is the identity. A Hermitian metric on an almost complex manifold (M, J) is a Riemannian metric that is invariant by J,

$$g(JX, JY) = g(X, Y).$$

Noting that J is negative self-adjoint with respect to g, i. e. $g(X, JY) = -g(JX, Y)$. We can define fundamental 2-form as

$$\Omega(X, Y) = g(X, JY).$$

An almost complex metric structure is said to be:

1. Almost Kählerian if $d\Omega = 0$;
2. Kählerian if $\nabla J = 0$;
3. Nearly Kählerian if $(\nabla_X J)(Y) + (\nabla_Y J)(X) = 0$;
4. Semi Kählerian if $\delta\Omega = 0$.

3. Lifted Almost Complex Structure

Let $(M, \eta, \xi, \varphi, g)$ be an almost contact metric structure and suppose we adopt Levi-Civita connection on it. Let $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ be a basis for TM. We can define an almost complex structure on basis elements on TM:

$$J(\frac{\delta}{\delta x^i}) = \varphi_i^j \frac{\delta}{\delta x^j} - \eta_i \xi^j \frac{\partial}{\partial y^j},$$
$$J(\frac{\partial}{\partial y^i}) = \eta_i \xi^j \frac{\delta}{\delta x^j} + \varphi_i^j \frac{\partial}{\partial y^j}.$$

Let g be a Riemannian metric for M, the metric g^s is defined by

$$g^s = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j.$$

It is easy to see that the following holds

$$\nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma_{ij}^h \frac{\delta}{\delta x^h} - \frac{1}{2} R_{ijk}{}^h y^k \frac{\partial}{\partial y^h},$$
$$\nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = -\frac{1}{2} R^h{}_{ikj} y^k \frac{\delta}{\delta x^h} + \Gamma_{ij}^h \frac{\partial}{\partial y^h},$$
$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = -\frac{1}{2} R^h{}_{jki} y^k \frac{\delta}{\delta x^h},$$
$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0.$$

Theorem 3.1. *Let J be a lifted almost complex structure on TM. Then (TM, J) is Kählerian if and only if (M, η, ξ, φ) is cosymplectic and $R = 0$.*

Proof. Let (M, J) be Kählerian. We calculate (∇J) on basis elements,

$$\begin{aligned}(\nabla_{\frac{\delta}{\delta x^i}} J)(\frac{\delta}{\delta x^j}) = & \frac{\partial}{\partial x^i}(\varphi_j^h) \frac{\delta}{\delta x^h} + \varphi_j^k \{\Gamma_{ik}^h \frac{\delta}{\delta x^h} - \frac{1}{2} R_{ikl}{}^h y^l \frac{\partial}{\partial y^h}\} - \frac{\partial}{\partial x^i}(\eta_j \xi^h) \frac{\partial}{\partial y^h} \\ & - \eta_j \xi^k \{-\frac{1}{2} R^h{}_{ilk} y^l \frac{\delta}{\delta x^h} + \Gamma_{ik}^h \frac{\partial}{\partial y^h}\} - \Gamma_{ij}^k \{\varphi_k^h \frac{\delta}{\delta x^h} - \eta_k \xi^h \frac{\partial}{\partial y^h} \\ & + \frac{1}{2} R_{ijl}{}^k y^l \{\eta_k \xi^h \frac{\delta}{\delta x^h} + \varphi_k^h \frac{\partial}{\partial y^h}\}.\end{aligned}$$

$$\begin{aligned}(\nabla_{\frac{\delta}{\delta x^i}} J)(\frac{\partial}{\partial x^j}) = & \frac{\partial}{\partial x^i}(\eta_j \xi^h) \frac{\delta}{\delta x^h} + \eta_j \xi^k \{\Gamma_{ik}^h \frac{\delta}{\delta x^h} - \frac{1}{2} R_{ikl}{}^h y^l \frac{\partial}{\partial y^h}\} + \frac{\partial}{\partial x^i}(\varphi_j^h \frac{\partial}{\partial y^h} \\ & + \varphi_j^k \{-\frac{1}{2} R_{ilk}^h y^l \frac{\delta}{\delta x^h} + \Gamma_{ik}^h \frac{\partial}{\partial y^h}\} + \frac{1}{2} R_{ilj}^k y^l \{\varphi_k^h \frac{\delta}{\delta x^h} - \eta_k \xi^h \frac{\partial}{\partial y^h}\} - \Gamma_{ij}^k \{\eta_k \xi^h \frac{\delta}{\delta x^h} + \varphi_k^h \frac{\partial}{\partial y^h}\}.\end{aligned}$$

$$(\nabla_{\frac{\partial}{\partial y^i}} J)(\frac{\delta}{\delta x^j}) = -\frac{1}{2}\varphi^k_j R^h_{kli} y^l \frac{\delta}{\delta x^h} + \frac{1}{2} R^k_{jli} y^l \{\varphi^h_k \frac{\delta}{\delta x^h} - \eta_k \xi^h \frac{\partial}{\partial y^h}\}$$
$$(\nabla_{\frac{\partial}{\partial y^i}} J)(\frac{\partial}{\partial x^j}) = \eta_j \xi^k (-\frac{1}{2} R^h_{kli} y^l) \frac{\delta}{\delta x^h}.$$

By above equations we have

$$\varphi^h_{j|i} + \frac{1}{2}\{R_{ijl}{}^k \eta_k \xi^h + \eta_j \xi^k R^h{}_{ilk}\} y^l = 0, \tag{1}$$
$$(\eta_j \xi^h)_{|i} + \frac{1}{2}\{\varphi^k_j R_{ikl}{}^h - \varphi^h_k R_{ijl}{}^k\} y^l = 0, \tag{2}$$
$$(\eta_j \xi^h)_{|i} + \frac{1}{2}\{\varphi^h_k R^k{}_{ilj} - \varphi^k_j R^h{}_{ilk}\} y^l = 0, \tag{3}$$
$$\varphi^h_{j|i} - \frac{1}{2}\{\eta_j \xi^k R_{ikl}{}^h + \eta_k \xi^h R^k{}_{ilj}\} y^l = 0,$$
$$(\varphi^k_j R^h{}_{kli} - \varphi^h_k R^k{}_{jli}) y^l = 0,$$
$$\eta_k \xi^h R^k{}_{jli} y^l = 0.$$

By (1) and (2) we have $\varphi^h_{j|i} = 0$ and $(\eta_j \xi^h)_{|i} = 0$, which means M is cosymplectic.
By (2) we have $R(X, \varphi Y)Z = \varphi(R(X, Y)Z)$ and by (3) we have $\varphi(R(X, Y)Z) = R(\varphi X, Y)Z)$. On the other hand, for a cosymplectic manifold we have $R(\varphi X, Y)Z = -R(X, \varphi Y)Z$, which is contradiction Thus we get $R = 0$. □

Theorem 3.2. *Let J be a lifted almost complex structure on TM. Then (TM, J) is nearly Kählerian if and only if (M, η, ξ, φ) is cosymplectic and $R = 0$.*

Proof. Let (TM, J) be a nearly Kählerian manifold, then by the same way we obtain the following

$$\varphi^h_{j|i} + \varphi^h_{i|j} + \frac{1}{2}\{\eta_j \xi^k R^h{}_{ilk} + \eta_i \xi^k R_{jlk}{}^h\} y^l = 0,$$
$$(\eta_i \xi^h)_{|j} + (\eta_j \xi^h)_{|i} - \frac{1}{2}\{\varphi^k_j R_{ikl}{}^h + \varphi^k_i R_{jkl}{}^h\} y^l = 0, \tag{4}$$
$$(\eta_j \xi^j)_{|i} - \frac{1}{2}\left\{\varphi^k_j R^h{}_{ilk} - 2\varphi^h_k R^k{}_{ilj} + \varphi^k_i R^h{}_{klj}\right\} y^l = 0, \tag{5}$$
$$\varphi^h_{j|i} - \frac{1}{2}\left\{\eta_j \xi^k R_{ikl}{}^h + 2\eta_k \xi^h R^k{}_{ilj}\right\} y^l = 0, \tag{6}$$
$$(\eta_j R^h{}_{kli} + \eta_i R^h{}_{klj}) \xi^k y^l = 0,$$

Then by (5) and (6), M is cosymplectic and from (4) we have $R(X, \varphi Y)Z = R(\varphi X, Y)Z$, but in a cosymplectic space $R(X, \varphi Y)Z = -R(\varphi X, Y)Z$ which is contradiction. Thus in a lifted nearly Kähler space $R = 0$. □

Theorem 3.3. *Let J be a lifted almost complex structure on TM. Then (TM, J) is Kählerian if and only if (M, η, ξ, φ) is nearly cosymplectic and*

$$\eta(R(X, Y)Z)\eta(K) - \eta(R(X, K)Z)\eta(Y) + \eta(R(Y, K)Z)\eta(X) = 0.$$

Proof. By definition of fundamental 2-form we have

$$\Omega(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = g(\frac{\delta}{\delta x^i}, J(\frac{\delta}{\delta x^j}) = \Phi_{ji},$$
$$\Omega(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = g(\frac{\delta}{\delta x^i}, J(\frac{\partial}{\partial y^j}) = \eta_i \eta_j,$$
$$\Omega(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = g(\frac{\partial}{\partial y^i}, J(\frac{\partial}{\partial y^j}) = \Phi_{ji}.$$

Thus for basis elements we have

$$\begin{aligned} d\Omega(\delta_i, \delta_j, \delta_k) &= \delta_i \Omega(\delta_j, \delta_k) - \delta_j \Omega(\delta_i, \delta_k) + \delta_k \Omega(\delta_i, \delta_j) - (R^h{}_{ijl}\Omega(\partial_h, \delta_k) - R^h{}_{ikl}\Omega(\partial_h, \delta_j) + R^h{}_{jkl}\Omega(\partial_h, \delta_i)) y^l \\ &= \frac{\partial}{\partial x^i}(\Phi_{kj}) - \frac{\partial}{\partial x^j}\Phi_{ki} + \frac{\partial}{\partial x^k}\Phi_{ji} + (R^h{}_{ijl}\eta_k - R^h{}_{ikl}\eta_j + R^h{}_{jkl}\eta_i)\eta_h y^l = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} d\Omega(\delta_i,\delta_j,\partial_k) &= \frac{\partial}{\partial x^i}(\eta_j\eta_k) - \frac{\partial}{\partial x^j}(\eta_i\eta_k) - \Gamma^h_{ik}\eta_j\eta_h - \Gamma^h_{jk}\eta_i\eta_h - R^h{}_{ijl}y^l = 0, \\ d\Omega(\delta_i,\partial_j,\partial_k) &= \frac{\partial}{\partial x^i}(\Phi_{kj}) - \Gamma^h_{ij}\Phi_{kh} - \Gamma^h_{ik}\Phi_{hj} = \Phi_{kj|i} = 0, \\ d\Omega(\partial_i,\partial_j,\partial_k) &= 0. \end{aligned} \tag{8}$$

Thus by (7) $d\Phi = 0$ and by (8) $d\eta = 0$ which means (M,η,ξ,φ) is almost cosymplectic. By the second part of (7) we have $\eta(R(X,Y)Z)\eta(K) - \eta(R(X,K)Z)\eta(Y) + \eta(R(Y,K)Z)\eta(X) = 0$. □

Theorem 3.4. *Let J be a lifted almost complex structure on TM. Then (TM,J) is semi Kählerian if and only if (M,η,ξ,φ) is semi cosymplectic and $R(X,Y)\varphi Z = 0$.*

Proof. Let (TM,J) be semi Kählerian. By direct calculation on basis elements we obtain

$$\begin{aligned} (\nabla\Omega)(\frac{\delta}{\delta x^k}) &= g^{ij}\{(\nabla_{\frac{\delta}{\delta x^i}}\Omega)(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}) + (\nabla_{\frac{\partial}{\partial y^i}}\Omega)(\frac{\partial}{\partial y^j},\frac{\delta}{\delta x^k})\} \\ &= g^{ij}\{\Phi_{kj|i}\} + \frac{1}{2}\xi^i\eta_h\{R_{ikl}{}^h - R_{ilk}{}^h\}y^l = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} (\nabla\Omega)(\frac{\partial}{\partial y^k}) &= g^{ij}\{(\nabla_{\frac{\delta}{\delta x^i}}\Omega)(\frac{\delta}{\delta x^j},\frac{\partial}{\partial x^k}) + (\nabla_{\frac{\partial}{\partial y^i}}\Omega)(\frac{\partial}{\partial y^j},\frac{\partial}{\partial x^k})\} \\ &= g^{ij}\{\eta_j\eta_k\}_{|i} + \frac{1}{2}R_{kli}{}^h\varphi^i_h y^l = 0. \end{aligned} \tag{10}$$

By first part of (9) and (10) we get (M,η,ξ,φ) is semi cosymplectic. From the second part of (9)) we obtain

$$\eta(R(\xi,Y)Z) = \eta(R(\xi,Z)Y),$$

and by (10) one concludes $R(X,Y)\varphi Z = 0$. □

Acknowledgment

The authors are grateful for Professor T. Tabatabeifar for her continuous help and encouragement.

References

[1] M. Ahmad and J.-B. Jun, *On semi-invariant submanifolds of a nearly Kenmotsu manifold with a quarter symmetric non-metric connection*, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math., 18 (2011), pp. 1–11.

[2] D. E. Blair and J. A. Oubiña, *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publ. Mat., 34 (1990), pp. 199–207.

[3] H. Faraji, B. Najafi Saghezchi, and T. Tabatabaeifar, *k-contact generalized square Finsler manifolds*, AUT J. Math. Com., 5 (2024), pp. 329–336.

[4] N. Heidari, N. H. P. Kashani, and B. Najafi, *Nearly Kähler and nearly Kenmotsu manifolds*, Turk. J. Math., 42 (2018), pp. 967–976.

[5] J. JAE-BOK, D. U. CHAND, and P. GOUTAM, *On Kenmotso manifolds*, J. Korean Math. Soc., 42 (2005), pp. 435–445.

[6] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. (2), 24 (1972), pp. 93–103.

[7] J.-S. Kim, X. Liu, and M. M. Tripathi, *On semi-invariant submanifolds of nearly trans-Sasakian manifolds*, Int. J. Pure Appl. Math. Sci., 1 (2004), pp. 15–34.

[8] H. Levy, *Symmetric tensors of the second order whose covariant derivatives vanish*, Ann. of Math. (2), 27 (1925), pp. 91–98.

[9] B. Najafi and N. Hosseinpour Kashani, *On nearly Kenmotsu manifolds*, Turkish J. Math., 37 (2013), pp. 1040–1047.

[10] A. Sardar, U. C. De, and B. Najafi, *3-dimensional f-Kenmotsu manifolds and solitons*, Int. J. Geom. Methods Mod. Phys., 19 (2022), pp. Paper No. 2250199, 15.

[11] S. Sasaki, *Almost contact manifolds, Part 2. Lecture Notes*, Mathematical Institute, Tohoku University, 1967.

[12] A. Shukla, *Nearly trans-Sasakian manifolds*, Kuwait J. Sci. Engrg., 23 (1996), pp. 139–144.

[13] T. Takahashi, *Sasakian ϕ-symmetric spaces*, Tohoku Math. J. (2), 29 (1977), pp. 91–113.

[14] S. Uddin, V. A. Khan, and K. A. Khan, *Warped product submanifolds of a Kenmotsu manifold*, Turkish J. Math., 36 (2012), pp. 319–330.

Please cite this article using:

Akbar Sadighi, Seyyed Mohammad Zamanzadeh, Almost complex structure over almost contact metric structures, AUT J. Math. Comput., 7(2) (2026) 183-188
https://doi.org/10.22060/AJMC.2024.23677.1285