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Original Article

Multivalued interpolative type contractions on partial metric spaces

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ABSTRACT: This article presents the interpolative fixed point theorem with reference to complete partial metric spaces, by taking the multi-valued contraction into account. In particular, the idea of multivalued interpolative Reich–Rus–Ćirić type contractions is introduced and criteria for the existence of fixed points' of such operators are established. A nontrivial example is provided to support the validity of the obtained results.

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1. Introduction

Banach (see [8]) established metric fixed point theory's foundation by proposing a prominent fixed point result. Banach observed that if a self-mapping Ω, defined on a complete metric space (χ, d), fulfills the contraction inequality, i.e., there exists a constant $\kappa \in [0,1)$ such that $d(\Omega t, \Omega u) \leq \kappa d(t,u)$ for all $t, u \in \chi$, then it possesses a unique fixed point. Kannan [11] proposed new fixed point and considered the contraction $d(\Omega t, \Omega u) \leq \kappa[d(t, \Omega t) + d(u, \Omega u)]$, for all $t, u \in \chi$, where $\kappa \in [0, \frac{1}{2})$. For a metric space (χ, d), the self-mapping $\Omega : \chi \longrightarrow \chi$ is said to be an interpolative Kannan type contraction, if there are constants $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that $d(\Omega t, \Omega u) \leq \lambda[d(t, \Omega t)]^{\alpha}.[d(u, \Omega u)]^{1-\alpha}$, for all $t, u \in \chi$ with $t \neq \Omega t$. Very recently, in Karapinar [12], the acclaimed theorem of Kannan was revisited by taking the interpolation theory into account. The main result in Karapinar [12] via an interpolative Kannan type contraction states that every interpolative Kannan type contraction possesses a unique fixed point. In addition,

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Rawat et al. [19] define interpolative enriched contractions of Kannan type, Hardy-Rogers type and Matkowski type, by enriching existing interpolative contractions, in the setting of convex metric space and prove existence of fixed points and approximation results using Krasnoselskij iteration. Moreover, Karapinar et al. [13] shows that the result of Karapinar [12] may not necessarily be unique (see [13, Example 1]). Furthermore, Karapinar et al. [14] investigate the necessary and sufficient conditions for the existence and uniqueness of the fixed point of Proinov type contractions, interpolative contractions, and ample spectrum contraction mappings in the settings of metric space. In addition, Reich, Rus and Ćirić [20, 21] combined and enhanced the Banach and Kannan fixed point theorems separately by defining a Reich–Rus–Ćirić contraction mapping, i.e., if Ω is a self-mapping defined on a complete metric space (χ, d) and $d(\Omega t, \Omega u) \leq \lambda[d(t,u) + d(t, \Omega t) + d(u, \Omega u)]$, for all $t, u \in \chi$, where $\lambda \in [0, \frac{1}{3})$, then Ω possesses a fixed point. Notice that several variations of Reich contractions can be stated. We may state the following: $d(\Omega t, \Omega u) \leq ld(t,u) + nd(t, \Omega t) + md(u, \Omega u)$, where $l, n, m \in (0, \infty)$ such that $0 \leq l + n + m < 1$. For more details on Interpolation theory (see [6, 15]). Also, [13] looked at the viability of the interpolation method for Reich contractions that Matthews [17] introduced in the context of partial metric spaces. Matthews [17] introduced the concept of a partial metric as a part of the study of denotational semantics of dataflow networks and gave a modified version of Banach's contraction principle, more suitable in this context. Many authors followed this idea and provided their contributions in that sense (see for example [3, 4]). Also, Karapinar et al. [16] investigate the existence and uniqueness of several distinct type contractive mapping in the context of complete partial metric space. Recently, Aydi et al. [5] introduced the idea of a partial Hausdorff metric and expanded the well-known Nadler's fixed point theorem to such spaces. On the other hand, Nadler [18] was the first who combined the ideas of multivalued mappings and contractions. Nadler [18] proved some remarkable results for multivalued contractions. Afterwards, several generalizations of Nadler's fixed point theorem, mainly by modifying the contractive condition, are obtained. In this vain, Sirajo et al. [22] introduced a new concept of multivalued contraction that was defined from a combination of Jaggi-type contraction, interpolative-type contraction and Pata-type inequality in the framework of metric space and analyzed the existence of fixed points for such mappings. Moreover, Gangwar et al. [10] introduce a new extension of S_b-metric spaces, called controlled S-metric spaces and establish some multivalued fixed point results. Also, Binayak et al. [9] proved fixed point results of multivalued mappings satisfying hybrid rational Pata-type inequalities (see for example, Ravi et al. [1], Akbar and Mohammed [7], Sirajo et al. [23], and references mentioned therein). Interestingly, the theory of multivalued mappings has many applications in economics, convex optimizations, optimal control theory and differential inclusions.

2. Preliminary

We recollect some basic definitions and terminologies that are required throughout the manuscript. Let (χ, ρ) be a partial metric space, $CB_\rho(\chi)$ be a collection of non-empty closed and bounded subset of χ with respect to the partial metric ρ. For $M \in CB_\rho(\chi)$, we define $\rho(a, M) = \inf\{\rho(a,t) : t \in M\}$. For $A, B \in CB_\rho(\chi)$, set

$$\delta_\rho(A, B) = \sup_{a \in A} \rho(a, B),$$

and

$$\delta_\rho(B, A) = \sup_{b \in B} \rho(b, A).$$

Also, for $A, B \in CB_\rho(X)$, define $H_\rho(A, B) = \max\{\delta_\rho(B, A), \delta_\rho(A, B)\}$.

Definition 2.1. *Let χ be a non-empty set. A function $\rho : \chi \times \chi \longrightarrow [0, \infty)$ is said to be a partial metric, if the following conditions are fulfilled for each $t, u, v \in \chi$:*

(ρ 1) if $\rho(t,t) = \rho(u,u) = \rho(t,u)$ then $t = u$;

(ρ 2) $\rho(t,t) \leq \rho(t,u)$;

(ρ 3) $\rho(t,u) = \rho(u,t)$;

(ρ 4) $\rho(t,u) \leq \rho(t,v) + \rho(v,u) - \rho(v,v)$.

In this case, (χ, ρ) is said to be a partial metric space.

The function $d_\rho : \chi \times \chi \longrightarrow [0, \infty)$ defined as $d_\rho(t,u) = 2\rho(t,u) - \rho(t,t) - \rho(u,u)$ is a standard metric on χ. It is natural to define the basic topological concepts, in particular, convergence of a sequence, fundamental (Cauchy) sequence criteria, continuity of the mappings, and completeness of the topological space in the framework of partial metric spaces (see [2, 16, 17]).

Definition 2.2. *In the framework of a partial metric space (χ, ρ), we say that*

(i) A sequence $\{t_i\}$ converges to the limit t, if $\rho(t,t) = \lim_{i\to\infty} \rho(t, t_i)$;

(ii) A sequence $\{t_i\}$ is fundamental or Cauchy if $\lim_{i,j\to\infty} \rho(t_i, t_j)$ exists and is finite;

(iii) A partial metric space (χ, ρ) is complete if each fundamental sequence $\{t_i\}$ converges to a point $t \in \chi$ such that $\rho(t,t) = \lim_{i,j\to\infty} \rho(t_i, t_j)$;

(iv) A mapping $F : \chi \longrightarrow \chi$ is continuous at a point $t_0 \in \chi$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_\rho(t_0, d)) \subseteq B_\rho(Ft_0, \epsilon)$.

Definition 2.3 ([13, Definition 3]). *Let (χ, ρ) be a partial metric space. A mapping $f : \chi \longrightarrow \chi$ is called an interpolative Reich–Rus–Ćirić type contraction, if there exist $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that*

$$\rho(ft, fu) \leq \lambda[\rho(t, u)]^\beta.[\rho(t, ft)]^\alpha.[\rho(u, fu)]^{1-\alpha-\beta},$$

for all $t, u \in \chi \setminus Fix(\Omega)$.

Lemma 2.4 ([13, Lemma 1]). *Let d_ρ be the matching standard metric space on the set χ, and let (χ, ρ) be a partial metric on a non-empty set χ.*

(i) A sequence $\{t_i\}$ is fundamental in the framework of a partial metric (χ, ρ) if and only if it is a fundamental sequence in the setting of the corresponding standard metric space (χ, d_ρ).

(ii) A partial metric space (χ, ρ) is complete if and only if the corresponding standard metric space (χ, d_ρ) is complete. Moreover,

$$\lim_{n\to\infty} d_\rho(t, t_i) = 0 \Leftrightarrow \rho(t,t) = \lim_{i\to\infty} \rho(t, t_i) = \lim_{i,j\to\infty} \rho(t_i, t_j).$$

(iii) If $t_i \to v$ as $i \to \infty$ in a partial metric space (χ, ρ) with $\rho(v, v) = 0$, then we have $\lim_{i\to\infty} \rho(t_i, v) = \rho(v, u)$ for every $u \in \chi$.

Lemma 2.5 ([5, Lemma 3.1]). *Let (χ, ρ) be a partial metric space, $A, B \in CB^\rho(\chi)$ and $\varrho > 1$. For any $a \in A$, there exists a point $b \in B$ such that*

$$\rho(a, b) \leq \varrho H_\rho(A, B).$$

We observe from the literature that there is insufficient research done on the fixed point theorem of multivalued contraction employing interpolative type. This study uses interpolative Reich–Rus–Ćirić type contractive inequality to present new multivalued fixed point results based on the prior knowledge. As a result, we identify and examine a few exceptional examples of our findings in the context of single-valued mappings that enhance certain related ideas.

3. Main Results

Our idea of multi-valued interpolative contraction of the Reich–Rus–Ćirić type is defined in light of the definition of interpolative Reich–Rus–Ćirić type contractions [13].

Definition 3.1. *Let (χ, ρ) be a partial metric space. A mapping $\Omega : \chi \longrightarrow CB^\rho(X)$ is called a multi-valued interpolative Reich–Rus–Ćirić type contraction, if there exist $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that*

$$H_\rho(\Omega t, \Omega u) \leq \lambda[\rho(t, u)]^\beta.[\rho(t, \Omega t)]^\alpha.[\rho(u, \Omega u)]^{1-\alpha-\beta}, \tag{1}$$

for all $t, u \in \chi \setminus Fix(\Omega)$.

Theorem 3.2. *In a complete partial metric space (χ, ρ) if $\Omega : \chi \longrightarrow CB^\rho(\chi)$ is a multi-valued interpolative Reich–Rus–Ćirić type contraction, then Ω possesses a fixed point in χ.*

Proof. Assume $t_0 \in \chi$ be arbitrary and choose $t_1 \in \Omega t_0$. Since $\Omega t_0 \in CB^{\rho}(\chi)$ and $t_1 \in \Omega t_0$, so by Lemma 2.5, there is $t_2 \in \Omega t_1$ such that $\rho(t_1, t_2) \leq \varrho H_\rho(\Omega t_0, \Omega t_1)$. Also, since $\Omega t_1 \in CB^{\rho}(\chi)$ and $t_2 \in \Omega t_1$, then by Lemma 2.5, there is $t_3 \in \Omega t_2$ such that $\rho(t_2, t_3) \leq \varrho^2 H_\rho(\Omega t_1, \Omega t_2)$. Continuing in this manner, we have a sequence $\{t_i\}$ of points of χ such that $t_{i+1} \in \Omega t_i$ and $\rho(t_i, t_{i+1}) \leq \varrho^i H_\rho(\Omega t_{i-1}, \Omega t_i)$ for each i $\in \mathbb{N}$. Consequently, from (1), we have

$$\begin{aligned}
\rho(t_i, t_{i+1}) &\leq \varrho^i H_\rho(\Omega t_{i-1}, \Omega t_i) \\
&\leq \varrho^i \lambda [\rho(t_{i-1}, t_i)]^\beta \cdot [\rho(t_{i-1}, \Omega t_{i-1})]^\alpha \cdot [\rho(t_i, \Omega t_i)]^{1-\alpha-\beta} \\
&\leq \varrho^i \lambda [\rho(t_{i-1}, t_i)]^\beta \cdot [\rho(t_{i-1}, t_i)]^\alpha \cdot [\rho(t_i, t_{i+1})]^{1-\alpha-\beta} \\
&\leq \varrho^i \lambda [\rho(t_{i-1}, t_i)]^{\beta+\alpha} \cdot [\rho(t_i, t_{i+1})]^{1-\alpha-\beta}. \qquad (2)
\end{aligned}$$

From inequality (2), we have

$$[\rho(t_i, t_{i+1})]^{\beta+\alpha} \leq \varrho^i \lambda [\rho(t_{i-1}, t_i)]^{\beta+\alpha}. \qquad (3)$$

Hence, inequality (3) gives

$$\begin{aligned}
\rho(t_i, t_{i+1}) &\leq \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\rho(t_{i-1}, t_i)] \\
&\leq \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\varrho^{i-1} H_\rho(\Omega t_{i-2}, \Omega t_{i-1})] \\
&\leq \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^\beta \cdot [\rho(t_{i-2}, \Omega t_{i-2})]^\alpha \cdot [\rho(t_{i-1}, \Omega t_{i-1})]^{1-\alpha-\beta}] \\
&\leq \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^\beta \cdot [\rho(t_{i-2}, t_{i-1})]^\alpha \cdot [\rho(t_{i-1}, t_i)]^{1-\alpha-\beta}] \\
&\leq \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^{\beta+\alpha} \cdot [\rho(t_{i-1}, t_i)]^{1-\alpha-\beta}]. \qquad (4)
\end{aligned}$$

But,

$$\rho(t_{i-1}, t_i) \leq \varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^{\beta+\alpha}.[\rho(t_{i-1}, t_i)]^{1-\beta-\alpha}. \qquad (5)$$

Thus, inequality (5) yield,

$$[\rho(t_{i-1}, t_i)]^{\beta+\alpha} \leq \varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^{\beta+\alpha}. \qquad (6)$$

Therefore, inequality (6) becomes

$$\rho(t_{i-1}, t_i) \leq \varrho^{\frac{i-1}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\rho(t_{i-2}, t_{i-1})]. \qquad (7)$$

So, from inequalities (4) and (7), we have

$$\begin{aligned}
\rho(t_i, t_{i+1}) &\leq \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^{\beta+\alpha}.[\rho(t_{i-1}, t_i)]^{1-\alpha-\beta}] \\
&\leq \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^{\beta+\alpha}.[\varrho^{\frac{i-1}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\rho(t_{i-2}, t_{i-1})]]^{1-\alpha-\beta}] \\
&= \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\varrho^{i-1} \lambda].[\varrho^{\frac{i-1}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}}]^{1-\alpha-\beta} \rho(t_{i-2}, t_{i-1})] \\
&= \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\varrho^{i-1}.\varrho^{(\frac{i-1}{\beta+\alpha})(1-\alpha-\beta)}].[\lambda.\lambda^{(\frac{1}{\beta+\alpha})(1-\alpha-\beta)}] \rho(t_{i-2}, t_{i-1})] \\
&= \varrho^{\frac{i}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} [\varrho^{\frac{i-1}{\beta+\alpha}} \lambda^{\frac{1}{\beta+\alpha}} (\rho(t_{i-2}, t_{i-1}))] \\
&= \varrho^{(\frac{2i-1}{\beta+\alpha})} \lambda^{(\frac{2}{\beta+\alpha})} [\rho(t_{i-2}, t_{i-1})] \\
&\leq \dots \\
&\leq \varrho^{[ni - \frac{1}{2}n(n-1)]k} \lambda^{nk} \rho(t_0, t_1),
\end{aligned}$$

where $k = \frac{1}{\beta+\alpha}$ and $i = 1, 2, \dots, n$.
Now, to show that (t_i) in χ is a fundamental sequence, let $j, i \in \mathbb{N}$, with $i < j$, then

$$\begin{aligned}
\rho(t_i, t_j) &\leq \rho(t_i, t_{i+1}) + \rho(t_{i+1}, t_{i+2}) + \dots + \rho(t_{j-1}, t_j) \\
&\leq \varrho^{[ni - \frac{1}{2}n(n-1)]k} \lambda^{nk} [\rho(t_0, t_1)] + \varrho^{[n(i+1) - \frac{1}{2}n(n-1)]k} \lambda^{nk} [\rho(t_0, t_1)] + \dots + \varrho^{[n(j-1) - \frac{1}{2}n(n-1)]k} \lambda^{nk} [\rho(t_0, t_1)] \\
&\leq [\varrho^{[ni - \frac{1}{2}n(n-1)]k} \lambda^{nk} + \varrho^{[n(i+1) - \frac{1}{2}n(n-1)]k} \lambda^{nk} + \dots + \varrho^{[n(j-1) - \frac{1}{2}n(n-1)]k} \lambda^{nk}] \rho(t_0, t_1) \\
&\leq [\varrho^{[ni - \frac{1}{2}n(n-1)]k} + \varrho^{[n(i+1) - \frac{1}{2}n(n-1)]k} + \dots + \varrho^{[n(j-1) - \frac{1}{2}n(n-1)]k}] \lambda^{nk} \rho(t_0, t_1) \\
&\leq [\varrho^{[ni - \frac{1}{2}n(n-1)]k} + \varrho^{[n(i+1) - \frac{1}{2}n(n-1)]k} + \dots + \varrho^{[n(j-1) - \frac{1}{2}n(n-1)]k}] \sum_{i=1}^{n} \lambda^{ik} \rho(t_0, t_1).
\end{aligned}$$

Letting $i \to \infty$ in the above inequality, shows that $\{t_i\}$ is a fundamental sequence. Hence, $\rho(t_i, t_j) \to 0$ as $i, j \to \infty$, that is $\{t_i\}$ is a fundamental sequence in (χ, ρ). By Lemma 2.4, $\{t_i\}$ is also a fundamental sequence in (χ, d_ρ). Also, since (χ, ρ) is complete, (χ, d_ρ) is also complete. So, there exists $t^* \in \chi$ such that $\rho(t^*, t^*) = \lim_{i\to\infty} \rho(t^*, t_i) = \lim_{i,j\to\infty} \rho(t_i, t_j) = 0$. This implies that, $\lim_{i\to\infty} d_\rho(t^*, t_i) = 0$. Now, we show that $t^* \in \Omega t^*$, then we have

$$\begin{aligned} \rho(t^*, \Omega t^*) &\le \rho(t^*, t_i) + \rho(t_i, \Omega t^*) - \rho(t_i, t_i) \\ &\le \rho(t^*, t_i) + \varrho^i H_\rho(\Omega t_{i-1}, \Omega t^*) - \rho(t_i, t_i) \\ &\le \rho(t^*, t_i) + \varrho^i \lambda[\rho(t_{i-1}, t^*)]^\beta.[\rho(t_{i-1}, \Omega t_{i-1})]^\alpha.[\rho(t^*, \Omega t^*)]^{1-\alpha-\beta} - \rho(t_i, t_i) \\ &= \rho(t^*, t_i) + \varrho^i \lambda[\rho(t_{i-1}, t^*)]^\beta.[\rho(t_{i-1}, t_i)]^\alpha.[\rho(t^*, \Omega t^*)]^{1-\alpha-\beta} - \rho(t_i, t_i). \end{aligned} \tag{8}$$

Letting $i \to \infty$ in (8), yield $\rho(t^*, \Omega t^*) \le 0$, so $\rho(t^*, \Omega t^*) = 0$, implies that $t^* \in \Omega t^*$. □

Corollary 3.3 ([13, Theorem 4]). *Let (χ, ρ) be a complete partial metric space and $f : \chi \longrightarrow X$ be an interpolative Reich–Rus–Ćirić type contractions, then f possesses a fixed point in χ.*

In what follows, a comparative example is constructed to support the hypotheses of Theorem 3.2.

Example 3.1. *Let $\chi = \{2i + 1 : i \in \{0, 1, 2, 3\}\}$ be a set equipped with the function $\rho : \chi \times \chi \longrightarrow \mathbb{R}$ defined by $\rho(t, u) = \max\{t, u\}$, for all $t, u \in \chi$. Then, (χ, ρ) is a complete partial metric space. Define a multivalued mapping $\Omega : \chi \longrightarrow CB(\chi)$ by*

$$\Omega t = \begin{cases} \{1, 3\} & \text{if } t \in \{1, 5\} \\ \{3, 5\} & \text{if } t \in \{3, 7\}. \end{cases}$$

Now, to verify the inequality (1), we see that for $t = u$ there is nothing to show. So, choose $\alpha = \frac{1}{2}$, $\beta = \frac{2}{5}$ and $\lambda = \frac{9}{10}$. Let $t, u \in \chi \setminus \mathrm{Fix}(\Omega)$. Then, $t, u \in \{5, 7\}$. Take $t = 5$ and $u = 7$, then

$$\begin{aligned} H_\rho(\Omega 5, \Omega 7) = H_\rho(\{1, 3\}, \{3, 5\}) &= 5 \\ &\le \lambda[\rho(5, 7)]^\beta[\rho(5, \Omega 5)]^\alpha[\rho(7, \Omega 7)]^{1-\beta-\alpha} \\ &\le \lambda[\rho(5, 7)]^\beta[\rho(5, \{1, 3\})]^\alpha[\rho(7, \{3, 5\})]^{1-\beta-\alpha} \\ &\le \lambda[\rho(5, 7)]^\beta[\rho(3, 5)]^\alpha[\rho(5, 7)]^{1-\beta-\alpha} \\ &\le \lambda[\rho(5, 7)]^{1-\alpha}[\rho(3, 5)]^\alpha \\ &\le \lambda[7]^{1-\alpha}[5]^\alpha. \end{aligned}$$

Hence, all the conditions of Theorem 3.2 are satisfied. We see that Ω has at least one fixed point. On the other hand, consider a self mapping f of χ, taking $t = 1$ and $u = 3$ such that $f1 = 1$ and $f3 = 3$. Then for $\lambda \in [0, \frac{1}{3})$, we obtain

$$\begin{aligned} \rho(f1, f3) = 3 &> \lambda[\rho(1, 3) + \rho(1, f1) + \rho(3, f3)] \\ &= \lambda[\rho(1, 3) + \rho(1, 1) + \rho(3, 3)] \\ &= \lambda[3 + 1 + 3] \\ &= 7\lambda, \end{aligned}$$

hence, it is clear that the contractive inequality (1) is not a Reich–Rus–Ćirić contraction. From Example 3.1 we see that the set of all fixed points of the mapping f is given by $F_f := \{1, 3\}$.

Proceeding as Theorem 3.2, we shall extend [13, Theorem 5] to multivalued settings.

Theorem 3.4. *Let (χ, ρ) be a complete partial metric space and $\Omega \colon \chi \longrightarrow CB^\rho(\chi)$ be such that $H_\rho(\Omega t, \Omega u) \le \lambda[\rho(t, \Omega t)]^\alpha.[\rho(u, \Omega u)]^{1-\alpha}$, for all $t, u \in \chi \setminus Fix(\Omega)$, where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$. Then Ω possesses a fixed point in χ.*

Proof. Let $t_0 \in \chi$ be arbitrary and choose $t_1 \in \Omega t_0$. Since $\Omega t_0 \in CB^\rho(\chi)$ and $t_1 \in \Omega t_0$, then by Lemma 2.5, there exists $t_2 \in \Omega t_1$ such that $\rho(t_1, t_2) \le \varrho H_\rho(\Omega t_0, \Omega t_1)$. Also, since $\Omega t_1 \in CB^\rho(\chi)$ and $t_2 \in \Omega t_1$, then by Lemma 2.5, there exists $t_3 \in \Omega t_2$ such that $\rho(t_2, t_3) \le \varrho^2 H_\rho(\Omega t_1, \Omega t_2)$. Continuing in this manner, we have a sequence $\{t_i\}$ of points of χ such that $t_{i+1} \in \Omega t_i$ and $\rho(t_i, t_{i+1}) \le \varrho^i H_\rho(\Omega t_{i-1}, \Omega t_i)$ for each $i \in \mathbb{N}$. Consequently, we have

$$\begin{aligned} \rho(t_i, t_{i+1}) &\le \varrho^i H_\rho(\Omega t_{i-1}, \Omega t_i) \\ &\le \varrho^i \lambda[\rho(t_{i-1}, \Omega t_{i-1})]^\alpha.[\rho(t_i, \Omega t_i)]^{1-\alpha} \\ &= \varrho^i \lambda[\rho(t_{i-1}, t_i)]^\alpha.[\rho(t_i, t_{i+1})]^{1-\alpha}. \end{aligned} \tag{9}$$

From inequality (9), we have

$$[\rho(t_i, t_{i+1})]^\alpha \quad \leq \quad \varrho^i \lambda [\rho(t_{i-1}, t_i)]^\alpha. \tag{10}$$

Hence, inequality (10) gives

$$\begin{aligned} \rho(t_i, t_{i+1}) \quad &\leq \quad \varrho^{\frac{i}{\alpha}} \lambda^{\frac{1}{\alpha}} [\rho(t_{i-1}, t_i)] \\ &\leq \quad \varrho^{\frac{i}{\alpha}} \lambda^{\frac{1}{\alpha}} [\varrho^{i-1} H_\rho(\Omega t_{i-2}, \Omega t_{i-1})] \\ &\leq \quad \varrho^{\frac{i}{\alpha}} \lambda^{\frac{1}{\alpha}} [\varrho^{i-1} \lambda [\rho(t_{i-2}, \Omega t_{i-2})]^\alpha . [\rho(t_{i-1}, \Omega t_{i-1})]^{1-\alpha}] \\ &= \quad \varrho^{\frac{i}{\alpha}} \lambda^{\frac{1}{\alpha}} [\varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^\alpha . [\rho(t_{i-1}, t_i)]^{1-\alpha}]. \end{aligned} \tag{11}$$

As,

$$\rho(t_{i-1}, t_i) \quad \leq \quad \varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^\alpha . [\rho(t_{i-1}, t_i)]^{1-\alpha}, \tag{12}$$

inequality (12) yields,

$$[\rho(t_{i-1}, t_i)]^\alpha \quad \leq \quad \varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^\alpha. \tag{13}$$

Therefore, inequality (13) becomes

$$\rho(t_{i-1}, t_i) \quad \leq \quad \varrho^{\frac{i-1}{\alpha}} \lambda^{\frac{1}{\alpha}} [\rho(t_{i-2}, t_{i-1})], \tag{14}$$

and then, from inequalities (11) and (14), we have

$$\begin{aligned} \rho(t_i, t_{i+1}) &\leq \varrho^{\frac{i}{\alpha}} \lambda^{\frac{1}{\alpha}} [\varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^\alpha . [\rho(t_{i-1}, t_i)]^{1-\alpha}] \\ &\leq \varrho^{\frac{i}{\alpha}} \lambda^{\frac{1}{\alpha}} [\varrho^{i-1} \lambda [\rho(t_{i-2}, t_{i-1})]^\alpha . [\varrho^{\frac{i-1}{\alpha}} \lambda^{\frac{1}{\alpha}} [\rho(t_{i-2}, t_{i-1})]]^{1-\alpha}] \\ &= \varrho^{\frac{i}{\alpha}} \lambda^{\frac{1}{\alpha}} [\varrho^{i-1} \lambda] . [\varrho^{\frac{i-1}{\alpha}} \lambda^{\frac{1}{\alpha}}]^{1-\alpha} \rho(t_{i-2}, t_{i-1}) \\ &= \varrho^{\frac{i}{\alpha}} \lambda^{\frac{1}{\alpha}} [\varrho^{(i-1)} . \varrho^{(\frac{i-1}{\alpha})(1-\alpha)}] . [\lambda . \lambda^{(\frac{1}{\alpha})(1-\alpha)}] \rho(t_{i-2}, t_{i-1}) \\ &= \varrho^{\frac{i}{\alpha}} \lambda^{\frac{1}{\alpha}} [\varrho^{\frac{i-1}{\alpha}} \lambda^{\frac{1}{\alpha}} (\rho(t_{i-2}, t_{i-1}))] \\ &= \varrho^{(\frac{2i-1}{\alpha})} \lambda^{(\frac{2}{\alpha})} [\rho(t_{i-2}, t_{i-1})] \\ &\leq \dots \\ &\leq \varrho^{[ni - \frac{1}{2} n(n-1)]k} \lambda^{nk} \rho(t_0, t_1), \end{aligned}$$

where $k = \frac{1}{\alpha}$ and $i = 1, 2, \dots, n$.
Now, to show that (t_i) in χ is a fundamental sequence, let $j, i \in \mathbb{N}$, with $i < j$, then

$$\begin{aligned} \rho(t_i, t_j) &\leq \rho(t_i, t_{i+1}) + \rho(t_{i+1}, t_{i+2}) + \dots + \rho(t_{j-1}, t_j) \\ &\leq \varrho^{[ni - \frac{1}{2} n(n-1)]k} \lambda^{nk} [\rho(t_0, t_1)] + \varrho^{[n(i+1) - \frac{1}{2} n(n-1)]k} \lambda^{nk} [\rho(t_0, t_1)] + \dots + \varrho^{[n(j-1) - \frac{1}{2} n(n-1)]k} \lambda^{nk} [\rho(t_0, t_1)] \\ &\leq [\varrho^{[ni - \frac{1}{2} n(n-1)]k} \lambda^{nk} + \varrho^{[n(i+1) - \frac{1}{2} n(n-1)]k} \lambda^{nk} + \dots + \varrho^{[n(j-1) - \frac{1}{2} n(n-1)]k} \lambda^{nk}] \rho(t_0, t_1) \\ &\leq [\varrho^{[ni - \frac{1}{2} n(n-1)]k} + \varrho^{[n(i+1) - \frac{1}{2} n(n-1)]k} + \dots + \varrho^{[n(j-1) - \frac{1}{2} n(n-1)]k}] \lambda^{nk} \rho(t_0, t_1) \\ &\leq [\varrho^{[ni - \frac{1}{2} n(n-1)]k} + \varrho^{[n(i+1) - \frac{1}{2} n(n-1)]k} + \dots + \varrho^{[n(j-1) - \frac{1}{2} n(n-1)]k}] \sum_{i=1}^{n} \lambda^{ik} \rho(t_0, t_1). \end{aligned}$$

Letting $i \to \infty$ in the above inequality, shows that $\{t_i\}$ is a fundamental sequence. Hence, $\rho(t_i, t_j) \to 0$ as $i, j \to \infty$, that is the sequence $\{t_j\}$ is a fundamental sequence in (χ, ρ). By Lemma 2.4, $\{t_i\}$ is also a fundamental sequence in (χ, d_ρ). Also, since (χ, ρ) is complete, (χ, d_ρ) is also complete. So, there exists $t^* \in \chi$ such that $\rho(t^*, t^*) = \lim_{i \to \infty} \rho(t^*, t_i) = \lim_{i,j \to \infty} \rho(t_i, t_j) = 0$. This implies that, $\lim_{i \to \infty} d_\rho(t^*, t_i) = 0$. Now, we show that $t^* \in \Omega t^*$. For this aim, we have

$$\begin{aligned} \rho(t^*, \Omega t^*) &\leq \rho(t^*, t_i) + \rho(t_i, \Omega t^*) - \rho(t_i, t_i) \\ &\leq \rho(t^*, t_i) + \varrho^i H_\rho(\Omega t_{i+1}, \Omega t^*) - \rho(t_i, t_i) \\ &\leq \rho(t^*, t_i) + \varrho^i \lambda [\rho(t_{i-1}, \Omega t_{i+1})]^\alpha \cdot [\rho(t^*, \Omega t^*)]^{1-\alpha} - \rho(t_i, t_i) \\ &\leq \rho(t^*, t_i) + \varrho^i \lambda [\rho(t_{i-1}, t_i)]^\alpha \cdot [\rho(t^*, \Omega t^*)]^{1-\alpha} - \rho(t_i, t_i). \end{aligned} \tag{15}$$

Letting $i \longrightarrow \infty$ in (15), yields $\rho(t^*, \Omega t^*) \leq 0$, so $\rho(t^*, \Omega t^*) = 0$, which implies that $t^* \in \Omega t^*$. □

Corollary 3.5 ([13, Theorem 5]). *Let (χ, ρ) be a complete partial metric space and let $f : \chi \longrightarrow \chi$ be an interpolative Reich–Rus–Ćirić type contractions type contraction, that is, $\rho(\Omega t, \Omega u) \leq \lambda[\rho(t, \Omega t)]^{\alpha} \cdot [\rho(u, \Omega u)]^{1-\alpha}$, for all $t, u \in \chi \setminus \mathrm{Fix}(\Omega)$, then Ω possess a fixed point in χ.*

4. Conclusion

In this paper, multi-valued interpolative Reich–Rus–Ćirić type contractions have been initiated as well as the corresponding fixed point theorems are proved (Theorem 3.2 and Theorem 3.4), an illustrative example (Example 3.1) supporting the hypotheses of the main result has been constructed. It has been observed that the idea studied herein improves a few corresponding results in the literature. Some of these special cases have been highlighted and discussed as corollaries.

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