



Short Contribution

A proof for a general form of the Serre-Swan theorem

Mohammad Bagher Asadi*, Zahra Hassanpour-Yakhdani

School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran

ABSTRACT: In this brief note, we present a proof for a general form of the Serre-Swan theorem.

Review History:

Received:02 January 2025

Revised:08 January 2025

Accepted:09 January 2025

Available Online:01 May 2026

Keywords:

C^* -algebra

Hilbert C^* -module

Serre-Swan theorem

MSC (2020):

46L08; 46L05; 42C15

1. Introduction

For a commutative C^* -algebra $C_0(\Omega)$, where Ω is a locally compact Hausdorff space, a general form of the Serre-Swan theorem [6] asserts that the category of Hilbert $C_0(\Omega)$ -modules is equivalent to the category of continuous fields of Hilbert spaces over Ω (see [4, 7]). For more information on the continuous field of Banach spaces, see [1, 3]. This approach can be used to study the structure of Hilbert C^* -modules over a commutative C^* -algebra. For example, in [5] this approach has been used to study the frame existence problem for a unital and commutative C^* -algebra. However, there is no basic and direct proof of the theorem in the literature. In this brief note, we shall state a basic and direct proof of the mentioned theorem.

2. A proof for a general form of the Serre-Swan theorem

Henceforth, we assume that Ω is a locally compact Hausdorff space and $C_0(\Omega)$ is the commutative C^* -algebra consisting of all complex-valued functions on Ω vanishing at infinity. In the following, we will recall the definition of continuous field of Hilbert spaces from [1].

*Corresponding author.

E-mail addresses: mb.asadi@ut.ac.ir (M. B. Asadi), zhasanpour91@gmail.com (Z. Hassanpour-Yakhdani)



Let $\{H_\omega\}_{\omega \in \Omega}$ be a family of Hilbert spaces. Every element of $\prod_{\omega \in \Omega} H_\omega$, that is, every function x defined in Ω such that $x(\omega) \in H_\omega$ for every $\omega \in \Omega$, is called a vector field.

Definition 2.1. Let $\{H_\omega\}_{\omega \in \Omega}$ be a family of Hilbert spaces. We denote by $C_0(\Omega) - \prod_{\omega \in \Omega} H_\omega$ the set

$$\{x \in \prod_{\omega \in \Omega} H_\omega : \omega \mapsto \|x(\omega)\| \in C_0(\Omega)\}.$$

For a subset Γ of $C_0(\Omega) - \prod_{\omega \in \Omega} H_\omega$, we say an element $x \in C_0(\Omega) - \prod_{\omega \in \Omega} H_\omega$ is locally a member of Γ and denote it by $x \in_{loc} \Gamma$, if for every $\omega \in \Omega$ and every $\epsilon > 0$, there is some $x' \in \Gamma$ such that $\|x(\omega') - x'(\omega')\| < \epsilon$, for all ω' in some neighborhood of ω .

Definition 2.2. Let Ω be a locally compact Hausdorff space. A continuous field Γ of Hilbert spaces over Ω is a family $\{H_\omega\}_{\omega \in \Omega}$ of Hilbert spaces, with a linear subspace $\Gamma \subseteq \prod_{\omega \in \Omega} H_\omega$ satisfying the following properties:

1. Γ is a subset of $C_0(\Omega) - \prod_{\omega \in \Omega} H_\omega$.
2. For every $\omega \in \Omega$, the set $\{x(\omega) : x \in \Gamma\}$ is equal to H_ω .
3. If $x \in_{loc} \Gamma$, then $x \in \Gamma$.

We denote a continuous field of Hilbert spaces over a locally compact Hausdorff space Ω by $(\{H_\omega\}_{\omega \in \Omega}, \Gamma)$. Indeed, $(\{H_\omega\}_{\omega \in \Omega})$ can be considered as a Hilbert $C_0(\Omega)$ -module equipped with the point-wise multiplication

$$(f.x)(\omega) = f(\omega)x(\omega),$$

for every $\omega \in \Omega$ and $C_0(\Omega)$ -valued inner product

$$\langle x, y \rangle(\omega) = \langle x(\omega), y(\omega) \rangle,$$

for all $f \in C_0(\Omega), x, y \in \Gamma$, and $\omega \in \Omega$ (for more details, we refer to [1]). Moreover, corresponding to every Hilbert $C_0(\Omega)$ -module \mathcal{E} , there is a unique continuous field of Hilbert spaces isomorphic to \mathcal{E} (see [?, 7]). However, there is no direct proof for the following fact in the literature, so we present its proof.

Theorem 2.3. Let \mathcal{E} be a Hilbert $C_0(\Omega)$ -module. There is a continuous field of Hilbert spaces $(\{H_\omega\}_{\omega \in \Omega}, \Gamma)$ isomorphic to \mathcal{E} .

Proof. Proof. For every $\omega \in \Omega$, let $N_\omega = \{x \in \mathcal{E} : \langle x, x \rangle = 0\}$. Since, for every $x \in \mathcal{E}$ there is some $y \in \mathcal{E}$ that $x = y\langle y, y \rangle^{\frac{1}{2}}$, one can easily see that N_ω is a closed right Hilbert submodule of \mathcal{E} . Consider the quotient space $H_\omega = \frac{\mathcal{E}}{N_\omega}$, equipped with inner product $[x + N_\omega, y + N_\omega]_\omega := \langle x, y \rangle(\omega)$. Indeed, $(H_\omega, [\cdot, \cdot]_\omega)$ is complete and hence a Hilbert space. Now, for every $x \in \mathcal{E}$, define $\hat{x} \in \prod_{\omega \in \Omega} H_\omega$ by

$$\hat{x}(\omega) = x + N_\omega.$$

Since $\|\hat{x}(\omega)\|^2 = \langle x, x \rangle(\omega)$, we have $\hat{x} \in C_0(\Omega) - \prod_{\omega \in \Omega} H_\omega$.

Now, we let $\Gamma = \{\hat{x} : x \in \mathcal{E}\}$ and show that Γ is a continuous field of Hilbert spaces. Indeed, Γ satisfies Properties (1) and (2) in Definition 2.2. Suppose that $\pi \in C_0 - \prod_{\omega \in \Omega} H_\omega$ and for every $\epsilon > 0$ and every $\omega' \in \Omega$, there is an open subset U of Ω consisting of ω' that for every $\omega \in U$, $\|\pi(\omega) - \hat{x}(\omega)\| < \epsilon$, for some $x \in \mathcal{E}$. Let $\epsilon > 0$. Since, $\pi \in C_0 - \prod_{\omega \in \Omega} H_\omega$, there is a compact set K , such that for every $\omega \in K^c$, $\|\pi(\omega)\| < \epsilon$. Now, by the assumption, for every $\omega' \in K$, there exist an open subset $U_{\omega'}$, containing ω' and some $x \in \mathcal{E}$ such that for every $\omega \in U_{\omega'}$, $\|\pi(\omega) - \hat{x}(\omega)\| < \epsilon$. Now, since the compact set K is a subset of $\bigcup_{\omega' \in K} U_{\omega'}$, there exist $\omega_1, \omega_2, \dots, \omega_n \in K$, $x_1, x_2, \dots, x_n \in \mathcal{E}$ and open subsets U_1, \dots, U_n containing $\omega_1, \dots, \omega_n$ such that $K \subseteq \bigcup_{i=1}^n U_i$ and for every $\omega \in U_i$, $\|\pi(\omega) - \hat{x}_i(\omega)\| < \epsilon$. Now, the partition of unity theorem yields that for the finite family of open sets $\{U_i : 1 \leq i \leq n\}$ there exists a finite family $\{f_i : \Omega \rightarrow [0, 1] \mid 1 \leq i \leq n\}$ of continuous functions, such that for every $1 \leq i \leq n$, $Supp f_i \subseteq U_i$ and $\sum_{i=1}^n f_i(\omega) = 1$, for all $\omega \in K$ and also $\sum_{i=1}^n f_i(\omega) \leq 1$, for all $\omega \in \bigcup_{i=1}^n U_i$ [2].

Now, considering $y = \sum_{i=1}^n x_i \cdot f_i$, we have $y \in \mathcal{E}$. Moreover, for every $\omega \in \Omega$,

$$\|\pi(\omega) - \hat{y}(\omega)\| < 2\epsilon.$$

In order to see that, for an arbitrary $\omega \in \Omega$, if $\omega \in K$, let $I_\omega = \{i \in \{1, \dots, n\} : \omega \in U_i\}$. The following holds:

$$\begin{aligned} \|\pi(\omega) - \hat{y}(\omega)\| &= \|\sum_{i=1}^n f_i(\omega)\pi(\omega) - \sum_{i=1}^n \hat{x}_i(\omega)f_i(\omega)\| \\ &\leq \sum_{i=1}^n f_i(\omega)\|\pi(\omega) - \hat{x}_i(\omega)\| \\ &= \sum_{i \in I_\omega} f_i(\omega)\|\pi(\omega) - \hat{x}_i(\omega)\| + \sum_{i \notin I_\omega} f_i(\omega)\|\pi(\omega) - \hat{x}_i(\omega)\| \\ &= \sum_{i \in I_\omega} f_i(\omega)\|\pi(\omega) - \hat{x}_i(\omega)\| < \epsilon \sum_{i \in I_\omega} f_i(\omega) \leq \epsilon. \end{aligned} \tag{1}$$

If $\omega \in (\bigcup_{i=1}^n U_i) \setminus K$, then

$$\|\pi(\omega) - \hat{y}(\omega)\| \leq \|\pi(\omega) - \sum_{i=1}^n f_i(\omega)\pi(\omega)\| + \|\sum_{i=1}^n f_i(\omega)\pi(\omega) - \sum_{i=1}^n \hat{x}_i(\omega)f_i(\omega)\| < 2\epsilon.$$

If $\omega \in (\bigcup_{i=1}^n U_i)^c$, then for every $i \in \{1, \dots, n\}$, we have $f_i(\omega) = 0$. Hence, $\hat{y}(\omega) = 0$, which yields that $\|\pi(\omega) - \hat{y}(\omega)\| = \|\pi(\omega)\| < \epsilon$. So, for every $\omega \in \Omega$, $\|\pi(\omega) - \hat{y}(\omega)\| < 2\epsilon$. In other words, $\|\pi - \hat{y}\| = \sup_{\omega \in \Omega} \|\pi(\omega) - \hat{y}(\omega)\| \leq 2\epsilon$. Hence, with choosing $\epsilon = \frac{1}{2k}$, for every $k \in \mathbb{N}$, there is some $x_k \in \mathcal{E}$ with $\|\pi - \hat{x}_k\| \leq \frac{1}{k}$. Since, for every $x \in \mathcal{E}$, $\|x\| = \|\hat{x}\|$, one can easily see that the sequence $\{x_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{E} and consequently, there is some $x \in \mathcal{E}$ such that $\{x_k\}_{k \in \mathbb{N}}$ converges to it. Moreover, $\pi = \hat{x}$ and so $\pi \in \Gamma$. □

Acknowledgement

The second author is supported by the National Elites Foundation.

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Mohammad Bagher Asadi, Zahra Hassanpour-Yakhdani, A proof for a general form of the Serre-Swan theorem, AUT J. Math. Comput., 7(2) (2026) 201-203
<https://doi.org/10.22060/AJMC.2025.23797.1308>

