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Near-Optimal Controls of a Fuel Cell Coupled with Reformer using Singular Perturbation methods

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ABSTRACT: A singularly perturbed model is proposed for a system comprised of a PEM Fuel Cell (PEM-FC) with Natural Gas Hydrogen Reformer (NG-HR). This eighteenth order system is decomposed into slow and fast lower order subsystems using singular perturbation techniques that provides tools for separation and order reduction. Then, three different types of controllers, namely an optimal full-order, a near-optimal composite controller based on the slow and the fast subsystems, and a near-optimal reduced-order controller based on the reduced-order model, are designed. The comparison of closed-loop responses of these three controllers shows that there are minimal degradations in the performance of the composite and the reduced order controllers.

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1- Introduction

Hydrogen is one of the most efficient carrier of clean energy in the world. By applying a fuel cell, the chemical energy produced by oxidation of hydrogen can be directly converted to electricity, without passing the state of heat and Carnot cycle [1].

Among different types of fuel cells, the Polymer Electrolyte Membrane Fuel Cell (PEM-FC) has been progressively improved in design [2]. Since it needs a reactant of hydrogen with high purity, there is a need for another process which produces hydrogen from an appropriate chemical substance. This is supplied by a Fuel Hydrogen Reformer. If the chemical substance is Natural Gas, the Fuel Hydrogen Reformer will be Natural Gas Hydrogen Reformer (NG-HR) [1].

Each of the two models of PEM-FC and NGHR has been separately studied in [3-6]. In [6], a single model has been developed by input coupling of PEM-FC and NG-HR linear models.

In this work, we study the single model in [6] and present its singular perturbation model. Then, by employing singular perturbation methods, a linear transformation is attained. We use this transformation to attain a new model in which the slow and fast modes are completely isolated from each other. The advantage of this decomposed model is that its slow and fast subsystems can be studied independently, thus, design of a controller can be done for each of them separately. Eventually, a composite of the two reduced-order controllers can be applied to the system. Moreover, one can design the controller only based on the reduced-order model. Although none of the two designs is claimed to be optimum, the responses closely resemble the behavior of the full-order optimal controller.

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In this paper, we design the optimal full-order controller, near-optimal composite controller (i.e., composition of fast and slow controllers), and the near-optimal reduced-order controller. The outputs, the control inputs, and the states are compared for the resultant close-loop systems.

This paper is organized as follows. Section 2 describes the mathematical model of the system. In Section 3, singular perturbation theory is presented. In Section 4, the designing process of three types of controllers is described based on singular perturbation model of the system. In Section 5, simulation results are given. Finally, we make the concluding remarks in Section 6.

2- Mathematical Mode

For the sake of brevity, we do not present the full description of PEM-FC and NG-HR models. We refer the readers to a detailed description in [3-6].

2- 1- Model of PEM-FC

The linearized state-space model of PEM-FC is given by [3-6]:

$$\frac{d\left(\delta x^{FC}(t)\right)}{dt} = A^{FC}\delta x^{FC}(t) + B^{FC}\delta u_{blower}(t)$$
$$\delta y^{FC}(t) = C^{FC}\delta x^{FC}(t) + D^{FC}\delta u_{blower}(t) \qquad (1)$$

where $x^{FC}(t)$, that denotes the vector of state-space states, $u_{blower}(t)$, that represents the vector of inputs, and $y^{FC}(t)$, that is the vector of outputs, are defined by:

$$x^{FC} = \left[m_{O_2}, m_{H_2}, m_{N_2}, \omega_{cp}, P_{sm}, m_{sm}, m_{H_2O_A}, P_{rm}\right]^T \qquad (2)$$

$$u^{FC}(t) = v_{cm}(t) \qquad (3)$$

$$y^{FC}(t) = \left[W_{cp}, P_{sm}, v_{st}\right]^T,\ W_{cp} = \omega_{CP}\left(\omega_{cp}, P_{sm}\right) \qquad (4)$$

Consider the following relation,

$$x^{FC} = x_{ss}^{FC} + \delta x^{FC}, \; u^{FC} = u_{ss}^{FC} + \delta u^{FC}, \; y^{FC} = y_{ss}^{FC} + \delta y^{FC} \tag{5}$$

where δ describes the variation of related vectors near the steady state, and the subscript ss stands for steady state. The linearization is done around the following steady state point [4]:

$$P_{net}^{ss} = 40\ kW, \; I^{ss} = 191\ A, \; v_{cm}^{ss} = 164\ V, \; \lambda_{O_2}^{ss} = W_{O_2}^{in} / W_{O_2}^{reacted} = 2 \tag{6}$$

Refer to Appendix I and Appendix II for more details.

2- 2- Model of NG-HR

The model of NG-HR is described as follows [3-6]:

$$\frac{d\left(\delta x^{FPS}\right)}{dt}(t) = A^{FPS}\delta x^{FPS}(t) + B^{FPS}\delta u^{FPS}(t)$$
$$\delta y^{FPS}(t) = C^{FPs}\delta x^{FPS}(t) + D^{FPS}\delta u^{FPS}(t) \tag{7}$$

where $x^{FPS}(t)$ is the vector of state-space states, $u^{FPS}(t)$ is the vector of input variables and $y^{FPS}(t)$ is the vector of the output variables as:

$$x^{FPS} = \left[T_{cpox}, P_{H_2}^{an}, P^{an}, P^{hex}, \omega_{blo}, P^{hds}, P_{CH_4}^{mix}, P_{air}^{mix}, P_{H_2}^{wrox}, P^{wrox}\right]^T \tag{8}$$

$$u^{FPS}(t) = \begin{bmatrix} u_{blower}(t) \\ u_{valve}(t) \end{bmatrix} \tag{9}$$

$$y^{FPS}(t) = \left[T_{cpox}(t) \quad y_{H_2}^{an}(t)\right]^T \tag{10}$$

Let δ describe the variation of related vectors near steady-stat. Then, we have:

$$x^{FPS} = x_{ss}^{FPS} + \delta x^{FPS}, \; u^{FPS} = u_{ss}^{FPS} + \delta u^{FPS}, \; y^{FPS} = y_{ss}^{FPS} + \delta y^{FPS} \tag{11}$$

The steady state value for T_{cpox} is 972 K and for $y_{H_2}^{an}$ is 8.8% [6, 7]. Refer to Appendix I and Appendix II for more details.

2- 3- Model of coupled system

With considering a common input between PEM-FC and NGHR, the two models defined by [1-7] are integrated into a single coupled eighteenth order model as follows [6]:

$$\begin{bmatrix} \delta\dot{x}^{FC}(t) \\ \delta\dot{x}^{FPS}(t) \end{bmatrix} = \begin{bmatrix} A^{FC} & 0 \\ 0 & A^{FPS} \end{bmatrix}\begin{bmatrix} \delta x^{FC}(t) \\ \delta x^{FPS}(t) \end{bmatrix} + \begin{bmatrix} B^{FC} & 0 \\ B_1^{FPS} & B_2^{FPS} \end{bmatrix}\begin{bmatrix} \delta u_{blower}(t) \\ \delta u_{valve}(t) \end{bmatrix}$$
$$\delta y(t) = \begin{bmatrix} \delta y^{FC}(t) \\ \delta y^{FPS}(t) \end{bmatrix} = \begin{bmatrix} C^{FC} & 0 \\ 0 & C^{FPS} \end{bmatrix}\begin{bmatrix} \delta x^{FC}(t) \\ \delta x^{FPS}(t) \end{bmatrix} + \begin{bmatrix} D^{FC} & 0 \\ D_1^{FPS} & D_2^{FPS} \end{bmatrix}\begin{bmatrix} \delta u_{blower}(t) \\ \delta u_{valve}(t) \end{bmatrix} \tag{12}$$

3- Singular Perturbation Model Of Two-time Scale Systems

A singular perturbation model of a linear time-invariant system with two-time scale is represented by:

$$\dot{x} = A_{11}x + A_{12}z + B_1u, \quad x(t_0) = x^0, \quad x \in R^n$$
$$\varepsilon\dot{z} = A_{21}x + A_{22}z + B_2u, \quad z(t_0) = z^0, \quad z \in R^m \tag{13}$$

where x and z are slow and fast state vectors of the system with dimensions n and m, respectively. u is control input vector with dimension r, and ε is a small positive parameter which describes small parasitic parameters in the model [8]. The system has n small eigenvalues near imaginary axis; and m eigenvalues with larger real parts. Therefore, the system has n dominant modes and m non-dominant modes. With ε=0, the order of model decreases from (n+m) to (n). The importance of using a singularly perturbed model of a system is that if by reducing the order of a system, some criteria of control design are not met, by recounting ε, the design can be improved [8].

3- 1- Definition of ε for a two-time scale system

Suppose the system (13) is stable, so all the eigenvalues of the matrix of system, A, where $A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{bmatrix}$, have negative real parts. By sorting its eigenvalues based on their real parts values in a large-to-small order, its vector of eigenvalues becomes:

$$e(A) = [\lambda_{s1}, \ldots, \lambda_{sn}, \lambda_{f1}, \ldots, \lambda_{fm}]^T, \; Re(\lambda_{fm}) < \ldots < Re(\lambda_{f1}) < Re(\lambda_{sn}) < \ldots < Re(\lambda_{s1}) < 0 \tag{14}$$

The eigenvectors associated with the eigenvalues of the slow and the fast subsystems are respectively:

$$e(A_s) = [\lambda_{s1}, \ldots, \lambda_{sn}]^T \tag{15}$$

and

$$e(A_f) = [\lambda_{f1}, \ldots, \lambda_{fm}]^T \tag{16}$$

If $|\lambda_{sn}| \ll |\lambda_{f1}|$, the system will have the properties of a system with two-time scale and ε can be defined as [9]:

$$\varepsilon = \frac{|Re(\lambda_{sn})|}{|Re(\lambda_{f1})|} \tag{17}$$

3- 2- Method of attaining decoupled model from singular perturbation model

Suppose that the model in the form of (13) is available. If the following condition satisfies:

$$\left\|\varepsilon A_{22}^{-1}\right\| \le \frac{1}{3}\left(\|A_0\| + \left\|\frac{A_{12}}{\varepsilon}\right\|\left\|A_{22}^{-1}A_{21}\right\|\right)^{-1}, \; A_0 = A_{11} - \frac{A_{12}}{\varepsilon}\left(A_{22}^{-1}A_{21}\right) \tag{18}$$

where A_{22} is a nonsingular matrix [8], and the operator ‖.‖

yields 2-norm of a matrix. A transformation matrix can be found such that the system in (13) will be transformed to the following format [8,9]:

$$\begin{bmatrix} \dot{x}_s \\ \dot{z}_f \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} x_s \\ z_f \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} u \qquad (19)$$

The system (19) is in an appropriate decoupled form. So, the state spaces associated with the slow states, x_s, and fast states, z_f, can be studied individually.

The condition in (18) is only an approximate condition for the convergence. Most of the time if the less conservative condition in [20] is satisfied, the decoupling can be performed [9].

$$\left\| \varepsilon A_{22}^{-1} \right\| \leq \left(\left\| A_0 \right\| + \left\| \frac{A_{12}}{\varepsilon} \right\| \left\| A_{22}^{-1} A_{21} \right\| \right)^{-1} \qquad (20)$$

3- 3- Attaining singular perturbation model of system with two-time scale

Suppose a state-space model of a linear time-invariant system is available and the system has two-time scale properties, i.e. ε in (17) has a small positive value; to achieve a singular perturbation model of the system with two-time scale, there are two steps as follows:

1) Estimating the number of slow and fast states

The model of the system, without considering the vector of inputs, is in the form of:

$$\dot{v} = Av \qquad (21)$$

For estimating the singular perturbation model, first we determine the number of slow and fast states. For this reason, a special method is used from the references [10, 11]. In this method, the variable n should be equal to r, which minimizes the cost function of V_r. The description of V_r is as follows:

$$V_r = \frac{U_r}{U_{r-1}}, \quad U_r = \frac{1 + \sqrt{\left(\left(n+m\right) - r\right)}}{\left|\lambda_{r+1}\right|} \qquad (22)$$
$$, 1 < r < \left(n+m\right) - 1$$

where n is the number of suspended slow states and (n+m) is the number of columns/rows of matrix A. After estimating n, the amount of ε will be calculated using equation (17).

2) Using Schur decomposition for attaining the standard singular perturbation model

Using Schur decomposition method [12, 13], a transformation can be calculated that transforms the matrix of system, A, to a matrix whose related sub-matrices satisfy the condition (18). [9] suggests a combination of two methods, permutation and scaling, to find a transformation matrix. Permutation rearranges the given states in a form which its first n states are corresponding to the slow states, and the next ones are related to the fast states. Scaling readjusts the units to reduce the norms of A_{22}^{-1}, A_0, A_{12} and L_0 as much as possible. This combination method has two weakness points. First, this method is based on trial and error, and it is not suitable for a system with large dimension. Second, the transformation matrix has only n+m nonzero elements of $(n+m)^2$ elements, thus, all the capabilities of transformation matrix are not used. If the matrix of a system has many off-diagonal nonzero elements, this method will not be appropriate. Alternatively, [8,7] extract transformation matrix using only the first two terms of exponential expansion of the matrix F(ε)=εA. In this paper, Schur decomposition is used instead, and the transformation matrix is calculated accordingly.

Using Schur decomposition, the matrix of system, A , can be transformed into a quasi-upper triangular one [12]. By another transformation which permutes the rows of the new matrix to an appropriate one, a standard form of singular perturbation can be attained, and then it can be decoupled and transformed into the form of (19). Theorem 1 and Theorem 2 describe this method in more details.

Theorem1. (A real Schur form of matrix A) [12]

Suppose that $a_1 \pm ib_1$, $a_2 \pm ib_2$, … and $a_p \pm ib_p$ are complex conjugated eigenvalues of matrix A and λ_{2p+1}, λ_{2p+2}, … and λ_q are its real eigenvalues with rank q. There is an orthogonal similarity transformation Q such that:

$$S \triangleq Q^T A Q = \begin{bmatrix} B_1 & & & & & \\ & \ddots & & & \times & \\ & & B_p & & & \\ & & & \lambda_{2p+1} & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_q \end{bmatrix} \qquad (23)$$

S is in quasi-upper triangular form, where $B_r(r=1,2,\ldots,p)$ is a 2×2 matrix with eigenvalues of $a_r \pm ib_r$.

Theorem 2. (block-diagonal decomposition from Schur decomposition) [13]

If the below transformation

$$Q^T A Q = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1q} \\ 0 & T_{22} & \cdots & T_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T_{qq} \end{bmatrix} \qquad (24)$$

is a Schur decomposition, $A \in C^{l \times l}$, and $\lambda(T_{ii}) \cap \lambda(T_{jj}) = \emptyset$, i≠j, then there is a nonsingular matrix $Y \in C^{l \times l}$ such as:

$$\left(QY\right)^{-1} A \left(QY\right) = diag\left(T_{11}, T_{22}, \ldots, T_{qq}\right) \qquad (25)$$

From Theorem 2, it can be inferred that if by Schur decomposition matrix A is transformed into matrix Â in the following format:

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \qquad (26)$$

and $\lambda(\hat{A}_{11}) \cap \lambda(\hat{A}_{22}) = \emptyset$, then there will be a transformation matrix Y which transforms Â to a block-diagonal form (19).

3- 4- A transformation matrix for simplification

From equation (12) the matrix of system is defined as:

$$A = \begin{bmatrix} A^{FC} & 0 \\ 0 & A^{FPS} \end{bmatrix} \tag{27}$$

Because of its special format, each of the sub-system FC and FPS can be decomposed into slow and fast sub-systems (for simplifying decomposition). Suppose that the zero-input system FC be given by:

$$\dot{v}_1 = A_{fc} v_1 \tag{28}$$

T_1 is the transformation matrix as:

$$\begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = T_1 v_1 \tag{29}$$

that transforms system (28) to the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} \Lambda_{1,fc} & 0 \\ 0 & \Lambda_{2,fc} \end{bmatrix} \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} \tag{30}$$

where x_1 and z_1 are the vectors of slow and fast states of the sub-system FC, respectively.
Let the system FPS be given by

$$\dot{v}_2 = A_{fps} v_2 \tag{31}$$

and T_2 be the transformation matrix as:

$$\begin{bmatrix} x_2 \\ z_2 \end{bmatrix} = T_2 v_2 \tag{32}$$

T_2 transforms system (31) to the following form:

$$\begin{bmatrix} \dot{x}_2 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{1,fps} & 0 \\ 0 & \Lambda_{2,fps} \end{bmatrix} \begin{bmatrix} x_2 \\ z_2 \end{bmatrix} \tag{33}$$

where x_2 and z_2 are the vectors of slow and fast states of sub-system FPS, respectively.
If we define a transformation T as:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \tag{34}$$

the system:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} A_{fc} & 0 \\ 0 & A_{fps} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{35}$$

will be transformed into the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{z}_1 \\ \dot{x}_2 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{fc,1} & 0 & 0 & 0 \\ 0 & \Lambda_{fc,2} & 0 & 0 \\ 0 & 0 & \Lambda_{fps,1} & 0 \\ 0 & 0 & 0 & \Lambda_{fps,2} \end{bmatrix} \begin{bmatrix} x_1 \\ z_1 \\ x_2 \\ z_2 \end{bmatrix} \tag{36}$$

By another transformation, T_3, which is defined by:

$$T_3 = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & I_3 & 0 & 0 \\ 0 & 0 & 0 & I_4 \end{bmatrix} \tag{37}$$

where I_i, i=1,…,4 are matrices with appropriate dimensions. Finally, the system will be transformed into the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{fc,1} & 0 & 0 & 0 \\ 0 & \Lambda_{fps,1} & 0 & 0 \\ 0 & 0 & \Lambda_{fc,2} & 0 \\ 0 & 0 & 0 & \Lambda_{fps,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} \tag{38}$$

Therefore, the slow and fast states, x and z, and the total transformation matrix, T_{total}, will be:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{39}$$

$$T_{total} = T_3 \times \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \tag{40}$$

4- Design Of Controllers

4- 1- Design of optimal controller for full-order system

The decoupled system achieved from applying appropriate transformations is given in (41):

$$\begin{cases} \dot{x}_s(t) = A_s x_s(t) + B_s u(t) \\ \dot{z}_f(t) = A_f z_f(t) + B_f u(t) \end{cases}, \quad \begin{cases} x_s(0) = x_0 \\ z_f(0) = z_0 \end{cases}$$
$$y(t) = C_s x_s(t) + C_f z_f(t) \tag{41}$$

The objective function is defined by:

$$J_{full-order} = \frac{1}{2}\int_0^\infty \left(y^T(t)y(t) + u^T(t)Ru(t)\right)dt, \; R > 0 \tag{42}$$

To achieve optimal control input based on objective function (42), the following algebraic Riccati equation shall be solved:

$$A_{new}^T \bar{K} + \bar{K}A_{new} - \bar{K}B_{new}R^{-1}B_{new}^T\bar{K} + C_{new}^T C_{new} = 0$$
$$A_{new} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix}, B_{new} = \begin{bmatrix} B_s \\ B_f \end{bmatrix}, C_{new} = \begin{bmatrix} C_s & C_f \end{bmatrix} \tag{43}$$

in order to reach the optimal control input [8]:

$$u = -R^{-1}B_{new}^T \bar{K} \begin{bmatrix} x_s(t) \\ z_f(t) \end{bmatrix} \tag{44}$$

4- 2- Design of near-optimal controller for reduced-order model

The near-optimal controller can be designed based on reduced-order model of system instead of the full-order one. For this reason, slow-subsystem is selected as reduced-order model. By setting $\dot{z}_f=0$ in (41), the reduced-order model is defined as follows:

$$\begin{aligned} \dot{x}_s(t) &= A_0 x_s(t) + B_0 u_s(t),\ x_s(0) = x_0 \\ y_s(t) &= C_0 x_s(t) + D_0 u_s(t) \end{aligned} \tag{45}$$

where

$$A_0 = A_s,\ B_0 = B_s,\ C_0 = C_s,\ D_0 = -C_f A_f^{-1} B_f \tag{46}$$

Since A_f is nonsingular, it contains only the fast modes, i.e. the largest eigenvalues of system, A_f^{-1}, exists.

The objective function is defined as follows [8]:

$$\begin{aligned} J_{reduced-order} = \frac{1}{2}\int_0^\infty (& x_s^T(t) C_0^T C_0 x_s(t) + 2u_s^T(t) D_0^T C_0 x_s(t) \\ & + u_s^T(t) R_0 u_s(t)) dt \end{aligned} \tag{47}$$

where:

$$R_0 = R + D_0^T D_0 \tag{48}$$

To obtain optimal control input with regards to (47), the below algebraic equation shall be solved:

$$A_0^T \bar{K} + \bar{K} A_0 - (\overline{K} B_0 + C_0^T D_0) R_0^{-1} (B_0^T \bar{K} + D_0^T C_0) + C_0^T C_0 = 0 \tag{49}$$

Then, the optimal control input will be achieved as [8]:

$$u_s(t) = -R_0^{-1}\left(B_0^T \bar{K} + D_0^T C_0\right) x_s(t) \tag{50}$$

4- 3- Design of near-optimal composite controller

Another way to design a near-optimal controller is based on both slow and fast isolated sub-systems. This controller approximates optimal controller better than near-optimal reduced-order one. In this way, the slow and fast subsystems are designed individually. Then, they are composed to form a composite controller [8]. This approach decreases the stiffness difficulties , because instead of designing a control input for a system with order n+m, two control inputs will be designed for two systems with orders n and m; therefore, the computational cost will dramatically decrease.

Consider the optimal control problem defined by equations (41) and (42); the following theorem addresses the estimation of the near-optimal composite controller.

Theorem 3. [8]

If G_2 is designed such that Re $\lambda(A_f+B_f G_2)<0$, then there will be an $\varepsilon^*>0$ such that if the composite control law:

$$u(t) = \left[\left(I_r + G_2 A_f^{-1} B_f\right) G_0\right] x(t) + G_2 z(t) \tag{51}$$

is applied to the system described by (41), the states and control inputs of the resulted closed loop system with any bounded initial conditions x_0 and z_0, for any finite t and all $\varepsilon\in]0,\varepsilon^*]$ are approximated by:

$$x(t) = x_s(t) + O(\varepsilon) \tag{52}$$

$$z(t) = -A_f^{-1}(B_s G_0) x_s(t) + z_f(t) + O(\varepsilon) \tag{53}$$

$$u(t) = u_s(t) + u_f(t) + O(\varepsilon) \tag{54}$$

where:

$$u_s(t) = G_0 x_s(t),\quad u_f(t) = G_2 z_f(t) \tag{55}$$

If, in addition G_0 is designed such that Re $\lambda(A_s+B_s G_0)<0$, there exists an $\varepsilon^*>0$ such that the closed loop system is asymptotically stable and equations (52)-(54) hold for all $\varepsilon\in]0,\varepsilon^*]$ and all $t\in[t_0,\infty[$.

By theorem 3, the problem of optimal control design can be divided into two completely isolated parts, one for the slow subsystem (C_s, A_s, B_s), and the other for the fast one (C_f, A_f, B_f). The near-optimal composite controller is achieved by controllers u_s and u_f using equation (54) [8]:

$$u_c(t) = u_s(t) + u_f(t) \tag{56}$$

For this reason, the problem (41) and (42) shall be decomposed into two separate problems. The output of system is also desired to be the sum of the outputs of slow and fast individual subsystems. For this reason, we rewrite equation (41) by using equations (52) and (53) as below:

$$\begin{aligned} y &= C_s x(t) + C_f z(t) \\ &= C_s\left[x_s(t) + O(\varepsilon)\right] + C_f\left[-A_f^{-1} B_s u_s(t) + z_f(t) + O(\varepsilon)\right] \\ &= \left(C_s x_s(t) - C_f A_f^{-1} B_s u_s(t)\right) + \left(C_f z_f(t)\right) + O(\varepsilon) \\ &= y_s(t) + y_f(t) + O(\varepsilon) \end{aligned} \tag{57}$$

Based on equation (57), the output of slow sub-system is defined by:

$$y_s(t) = C_s x_s(t) - C_f A_f^{-1} B_s u_s(t) \tag{58}$$

which is associated to the slow sub-system:

$$\dot{x}_s(t) = A_s x_s(t) + B_s u_s(t),\quad x_s(0) = x_0 \tag{59}$$

and the output of the fast sub-system is defined by:

$$y_f(t) = C_f z_f(t) \tag{60}$$

which is associated to the fast sub-system [8]:

$$\dot{z}_f(t) = A_f z_f(t) + B_f u_f(t), \tag{61}$$

where

$$u_f(t) = u(t) - u_s(t) \tag{62}$$

and

$$z_f(t) = z(t) - z_s(t),\quad z_s(t) = -A_f^{-1} B_f u_s(t) \tag{63}$$

Therefore, the initial value of z_f can be calculated based on (63) as below:

$$z_f(t_0) = z_0 - z_s(t_0) \tag{64}$$

The objective function for the slow sub-system is defined as:

$$\begin{aligned} J_{slow} &= \int_0^\infty \left(x_s^T C_s^T C_s x_s + 2u_s^T D_s^T C_s x_s + u_s^T\left(R + D_s^T D_s\right) u_s\right) dt, \\ R &> 0 \end{aligned} \tag{65}$$

Hence, the optimal controller for the slow sub-system, u_s, can be estimated by

$$u_s(\mathrm{t}) = -\left[R + D_s^T D_s\right]^{-1}\left(D_s^T C_s + B_s^T K_s\right)x_s(t) \quad (66)$$

where K_s is the solution of below algebraic Riccati equation [8]:

$$\begin{aligned} 0 = &-K_s\left(A_s - B_s\left[R + D_s^T D_s\right]^{-1} D_s^T C_s\right) \\ &-\left(A_s - B_s\left[R + D_s^T D_s\right]^{-1} D_s^T C_s\right)^T K_s \\ &+K_s B_s\left[R + D_s^T D_s\right]^{-1} B_s^T K_s \\ &-C_s^T\left(I - D_s\left[R + D_s^T D_s\right]^{-1} D_s^T\right)C_s \end{aligned} \quad (67)$$

The objective function for the fast sub-system is defined as:

$$J_{fast} = \frac{1}{2}\int_0^\infty \left(z_f^T C_f^T C_f z_f + u_f^T R u_f\right)dt, R > 0 \quad (68)$$

Thus, the optimal controller u_f for the fast sub-system is

$$u_f(\mathrm{t}) = -R^{-1}B_f^T K_f z_f(t) \quad (69)$$

where K_f is the solution of below algebraic equation [8]:

$$0 = -K_f A_f - A_f^T K_f + K_f B_f R^{-1} B_f^T K_f - C_f^T C_f \quad (70)$$

5- Simulation Results

5- 1- Finding number of slow states and ε

Using the method described in sub-section 3.C.1 and equation (22), the number of slow states becomes 9, i.e. n=9. In Figure 1, V_r has been plotted as a function of r, i.e. the number of suspended slow states. By setting r=9 , the pseudo minimum of V_r is resulted. Absolute minimum of V_r occurs in r=2. However, to have a good estimation of system, we select 9 as the number of slow states of system because the slow subsystem can be an approximation of the full-order system. From on equation (17), ε is achieved as:

$$\varepsilon = 0.2739 \quad (71)$$

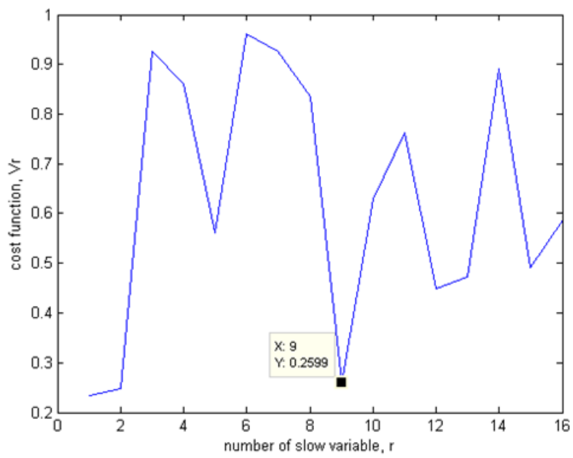


Fig. 1. Estimating the number of slow states

5- 2- To obtain decoupled singular perturbation model

Using methods given in sub-section 3.C.2 and 3.D, the decoupled singular perturbation model of system is achieved in the form of (41). The associated matrices are presented in Table 1.

5- 3- Results of applying optimal controller, near-optimal reduced and composite controllers

The matrix R in objective function (42) based on reference [6] is defined as:

$$R = 0.01\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (72)$$

The initial values of transformed state states (41) for full-order model is selected as:

$$\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = 0.1\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{18\times1} \quad (73)$$

Thus, for the reduced-order model (45) we have

$$x_0 = 0.1\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{9\times1} \quad (74)$$

For the design of composite controller, the initial values for states of the slow sub-system are similar to equation (74), but the initial values for states of fast sub-system are given by (61). With having identical initial conditions, the closeness of outputs under three types of controllers to that under the optima controller can be inferred by plotting all outputs in one diagram. Figure 2 shows these responses. As it can be observed from Figure 2, both responses associated to near-optimal reduced-order controller and near-optimal composite controller behave closely to the optimal full-order controller. The values of objective functions related to full-order system ($J_{full\text{-}order}$) and reduced order system ($J_{reduced\text{-}order}$) are attained as 1.2817 and 1.3601. The value of objective function for the composite controller ($J_{composite}$) is attained as 1.2841. This result is expected because the composite controller should be more accurate than the reduced-order one. This is also an expected result based on the reference [8] which asserts that:

$$J_{full-order} < J_{composite} < J_{reduced-order} \quad (75)$$

It can be realized from the amount of objective functions that the loss of performance compared with the optimal cost, with near-optimal composite controller, is less than 0.18% and with near-optimal reduced-order controller is less than 6.12%. Thus, using each of the near-optimal composite or reduced-order controllers results in a performance close to that of the full-order optimal controller. This also indicates that this model has singular perturbation properties with two-time scale.

Furthermore, the trajectories of the basic state space system under the optimal full-order, reduced-order, and composite controllers, described by (12) are plotted all in a single diagram to be compared with each other (Figure 3); also the comparison of related control inputs is presented in Figure 4. Figure 3 and Figure 4 also exhibit that near-optimal controllers are acceptable approximates for the optimal ones.

Table 1. matrices of decoupled system

A_s	$\begin{matrix} -1.4030 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.6738 & -1.6473 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8.3596 & -0.3087 & -2.9158 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.3333 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5.1226 & -1.4680 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57.4801 & 2.6790 & -2.7710 & 0.0756 & 0 & 0 \\ 0 & 0 & 0 & 64.5906 & 2.9102 & -3.9643 & -2.7710 & 0 & 0 \\ 0 & 0 & 0 & -43.1159 & -6.0750 & 0.8337 & 1.5273 & -0.3580 & 0 \\ 0 & 0 & 0 & -65.8496 & -15.2075 & -6.7233 & -8.5388 & 0.0286 & -0.0862 \end{matrix}$	$\frac{A_f}{\varepsilon}$	$\begin{matrix} -22.4029 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1181 & -18.2586 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.1614 & 20.9866 & -46.1771 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.2977 & -25.1165 & 39.5689 & -89.4854 & 0 & 0 & 0 & 0 & 0 \\ -431.7089 & 17.3312 & -17.4515 & -9.5467 & -219.6241 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12.1690 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -128.5911 & -89.1369 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.4631 & 0.1823 & -157.8979 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30.1116 & -32.5183 & -0.8270 & -660.6821 \end{matrix}$
B_s	$\begin{matrix} 0.0039 & 0 \\ 0.0976 & 0 \\ -0.3102 & 0 \\ -0.1200 & 0 \\ 0 & 0.1770 \\ 0 & -0.0214 \\ 0 & 0.0395 \\ 0 & 0.0009 \\ 0 & 0.0021 \end{matrix}$	B_f	$\begin{matrix} -0.0035 & 0 \\ -2.6051 & 0 \\ -2.2066 & 0 \\ -0.4811 & 0 \\ -0.0025 & 0 \\ -0.0788 & 0.2253 \\ -0.5270 & -0.7613 \\ -0.0017 & 0.0019 \\ 0.0165 & 0.1158 \end{matrix}$
C_s	$\begin{matrix} -15.6122 & 0.2106 & 6.4158 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3698 & -0.0072 & -0.1326 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14.8180 & 12.4493 & -0.6497 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1239 & 0.6616 & 0.9675 & 0.6094 & 0.1165 & 1.1066 \\ 0 & 0 & 0 & -0.1869 & -0.4375 & -0.7381 & 0.0094 & -0.9552 & -0.0070 \end{matrix}$	C_f	$\begin{matrix} 0.2118 & -0.4534 & -8.0858 & -6.2472 & 0 & 0 & 0 & 0 & 0 \\ -0.0023 & -0.0332 & 0.0477 & 0.0480 & 0 & 0 & 0 & 0 & 0 \\ -9.8329 & 0.9992 & -0.0766 & -1.9067 & -3.0379 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.4248 & 0.0352 & 0 & -0.0029 \\ 0 & 0 & 0 & 0 & 0 & 0.2798 & 0.0009 & -0.0768 & -0.0002 \end{matrix}$

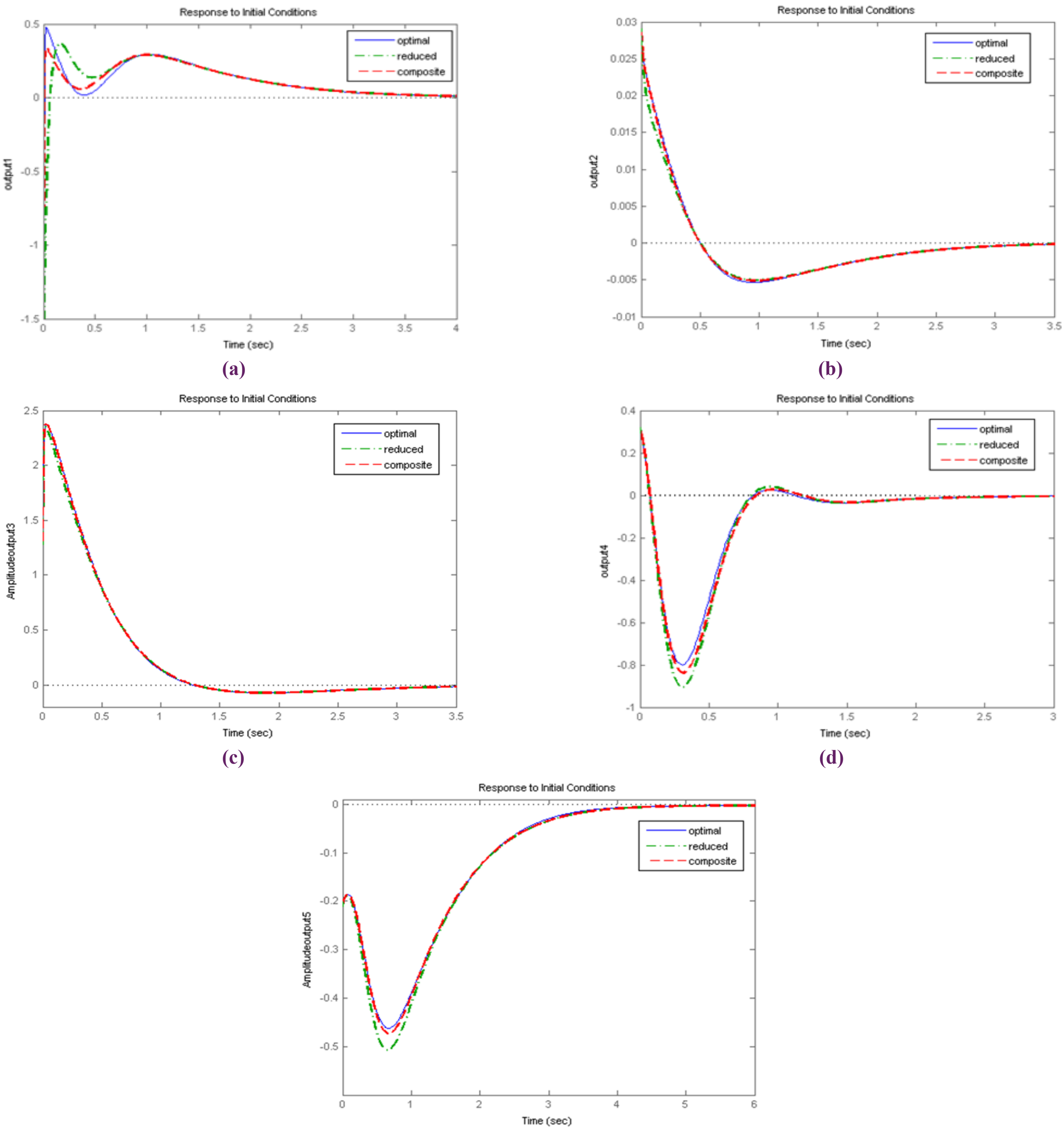


Fig. 2. output responses for different types of controllers (a) 1st output, (b) 2nd output,…, (e) 5th output

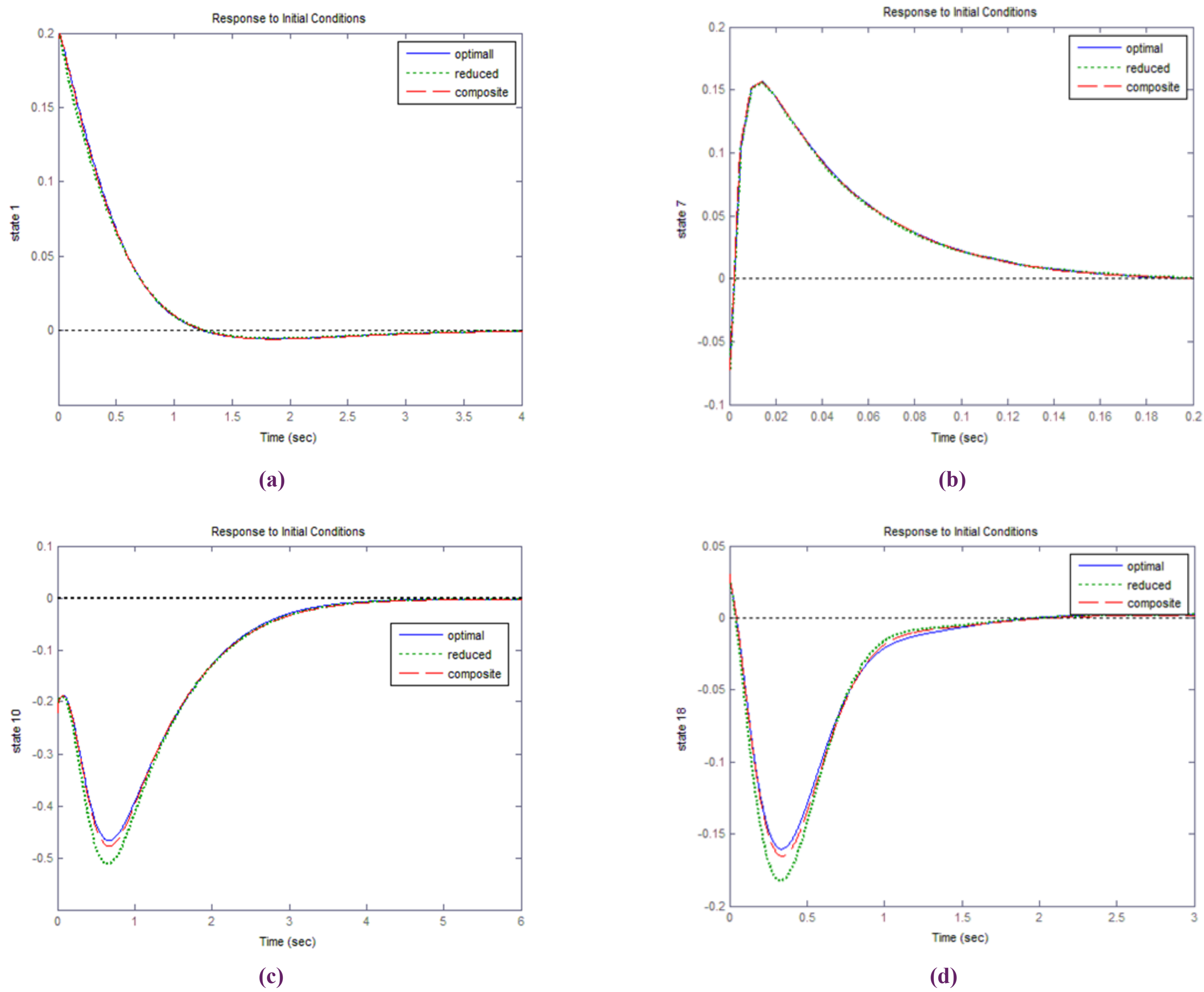
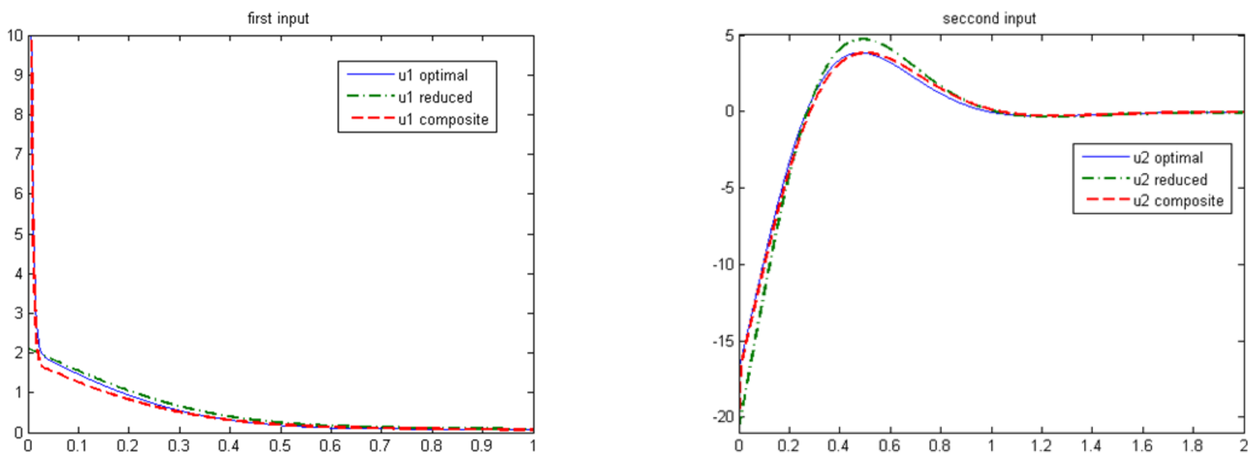


Fig. 3. trajectories of basic states for three types of optimal controllers; (a) 1st state, (b) 7th state, (c) 10th state, (d) 18th state.



(a) (b)

Fig. 4. comparison of three types of optimal control design; (a) 1st input, (b) 2nd input.

6- CONCLUSION

A singular perturbation model of the eighteenth order system of Polymer Electrolyte Membrane Fuel Cell coupled with Natural Gas Hydrogen Reformer with two-time scale was introduced in this paper. Applying the near-optimal controller, designed based on the reduced-order model, and slow and fast models resulted from the two-time scale model, indicates that all the simpler controllers have a performance near to optimal full-order controller. This shows the achieved model of two-time scale singularly perturbed is acceptable.

Appendix I: (matrices of original system) [6]

$$A^{FC}=\begin{bmatrix} -6.3091 & 0.0000 & -10.954 & 0.0000 & 83.7446 & 0.0000 & 0.0000 & 24.0587 \\ 0.0000 & -161.08 & 0.0000 & 0.0000 & 51.5292 & 0.0000 & -18.026 & 0.0000 \\ -18.786 & 0.0000 & -46.314 & 0.0000 & 275.659 & 0.0000 & 0.0000 & 158.374 \\ 0.0000 & 0.0000 & 0.0000 & -17.351 & 193.937 & 0.0000 & 0.0000 & 0.0000 \\ 1.2996 & 0.0000 & 2.9693 & 0.3977 & -38.702 & 0.1057 & 0.0000 & 0.0000 \\ 16.6424 & 0.0000 & 38.0252 & 5.0666 & -479.38 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -450.39 & 0.0000 & 0.0000 & 142.208 & 0.0000 & -80.947 & 0.0000 \\ 2.0226 & 0.0000 & 4.6212 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -51.211 \end{bmatrix}$$

$$B^{FC}=\begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \\ 3.9467 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}$$

$$C^{FC}=\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 5.0666 & -116.4500 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 12.9699 & 10.3235 & -0.5693 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$D^{FC}=\begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}$$

$$A^{FPS}=\begin{bmatrix} -0.074 & 0 & 0 & 0 & 0 & 0 & -3.53 & 1.0748 & 0 & 1e-6 \\ 0 & -1.468 & -253 & 0 & 0 & 0 & 0 & 0 & 2.5582 & 13.911 \\ 0 & 0 & -156 & 0 & 0 & 0 & 0 & 0 & 0 & 33.586 \\ 0 & 0 & 0 & -124.5 & 212.63 & 0 & 112.69 & 112.69 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.33 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -32.43 & 32.304 & 32.304 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 331.8 & -344 & -341 & 0 & 9.9042 \\ 0 & 0 & 0 & 221.97 & 0 & 0 & -253.2 & -254.9 & 0 & 32.526 \\ 0 & 0 & 2.0354 & 0 & 0 & 0 & 1.8309 & 1.214 & -0.358 & -3.304 \\ 0.0188 & 0 & 8.1642 & 0 & 0 & 0 & 5.6043 & 5.3994 & 0 & -13.61 \end{bmatrix}$$

$$B^{FPS}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.12 & 0 \\ 0 & 0.1834 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C^{FPS}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.994 & -0.088 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D^{FPS}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Appendix II: Naotations [6]

Nomenclature			
m_{O_2}	mass of oxygen	u_{blo}	air blower signal
m_{H_2}	mass of hydrogen	u_{valve}	valve blower signal
m_{N_2}	mass of nitrogen	T_{cpox}	catalyst temperature
ω_{cp}	compressor speed, rad/sec	$p^{an}{}_{H2}$	pressure of hydrogen in the anode
p_{sm}	pressure of gas in supply manifold	p^{an}	anode pressure
m_{sm}	mass of gas in supply manifold	p^{hex}	heat exchanger pressure
m_{H2OA}	mass of water in the anode channel	ω_{blo}	speed of the blower , rad/sec
p_{rm}	pressure in the return manifold	p^{hds}	pressure of hydro-desulfurizer
W_{cp}	compressor flow rate	$p^{mix}{}_{CH4}$	pressure of CH4 in the mixer
v_{st}	stack voltage	$p^{mix}{}_{air}$	pressure of air in the mixer
v_{cm}	compressor motor input voltage	$p^{wrox}{}_{H2}$	hydrogen pressure in water gas shift converter (WROX)
I_{st}	stack current	p^{wrox}	total pressure in WROX

REFERENCE

[1] Pukrushpan, J.T., modelling and control of fuel cell systems and fuel processor, Ph.D. Dissertation, Dept. Mechanical Engineering, University of Michigan, (2003).

[2] Pukrushpan, J.T., Stefanopoulou, A.G.,and Peng, H., Control of fuel cell breathing, IEEE Control Systems Magazine, vol. 24, no. 2, pp. 30-46 (2004).

[3] Pukrushpan, J., Stefanopoulou, A., and Peng, H., Control of fuel cell breathing: Initial Results on the oxygen starvation problem, Ann Arbor, National Science Foundation & Automotive Research Center of University of Michigan.

[4] Pukrushpan, J., Stefanopoulou, A., and Peng, H., Control of fuel cell power systems: principles, modeling, analysis, and feedback design, Springer (2004).

[5] Tsourapas, V., Stefanopoulou, A. G., and Sun, J., Model-based control of an integrated fuel cell and fuel processor with exhaust heat recirculation, IEEE Transactions on Control System Technology, vol.15, n0. 2, pp. 235-45 (2007).

[6] Kodra, K. and Gajic, Z., Order reduction via balancing and suboptimal control of a fuel cell-Reformer system, International Journal of Hydrogen energy, vol. 39, issue 5, pp. 2215-2223, (2014).

[7] Skataric, M. and Gajic, Z., Slow and fast dynamics of a natural gas hydrogen reformer, International Journal of Hydrogen Energy, vol. 38, issue 35, pp. 15173-9, (2013).

[8] Kokotovic, P., Khali, H.K., and O'Reilly, J., Singular Perturbation Methods in Control: Analysis and Design, London: Academic press, ch. 1,2,3,6 (1986).

[9] Naidu, D.S., Singular perturbation methodology in control systems, England: IEE control engineering series 34, Ch. 3, pp. 76-91 (1988).

[10] Rao, A. S., Lamba, S. S.,and Rao, S. V., comments on a note on selecting a low order system, IEEE Transaction Automatic Control, AC-24, pp.141-142 (1979).

[11] Fossard, A.J., Berthelot, M.,and Magni, J.F., On coherency-based decomposition Algorithms, Automatica, Vol. 19, No. 3, pp. 247-253 (1983).

[12] Datta, K.B., Matrix and Linear Algebra, New Delhi: Prentice Hall of India, pp. 541-2 (2004).

[13] Noble, B., and Daniel, J.W., Applied linear algebra, Englewood Cliffs, New Jersey: Prentice-Hall, pp.278-307 (1988).

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